An introduction to fractional calculus

Fundamental ideas and numerics

Fabio Durastante

Università di Pisa

✉ fabio.durastante@unipi.it

🌐 fdurastante.github.io

May 2, 2022

The original idea

The concept of differentiation and integration to noninteger order goes as far back as the concept we are used to work with. Leibniz mentions it in a letter to L'Hôspital in 1695:

*"John Bernoulli seems to have told you of my having mentioned to him a marvelous analogy which makes it possible to say in a way that successive differentials are in geometric progression. **One can ask what would be a differential having as its exponent a fraction**. You see that the result can be expressed by an infinite series. Although this seems removed from Geometry, which does not yet know of such fractional exponents, **it appears that one day these paradoxes will yield useful consequences, since there is hardly a paradox without utility**. Thoughts that mattered little in themselves may give occasion to more beautiful ones."*

(Leibniz, 1646-1716)

Who cares?

Derivatives of non integer order help

- modeling of *viscoelastic* phenomena, *e.g.*, (Bagley and Torvik 1986; Müller et al. 2011)
- restate fundamental model from physics [gravity (Giusti, Garrappa, and Vachon 2020), Schrödinger (Laskin 2002), waves (Luchko 2013), …],
- modeling of heterogeneous cardiac tissues (Cusimano et al. 2015),
- describing phenomena with *memory* and *non locality* aspects, *e.g.*, (Benzi et al. 2020; Riascos and Mateos 2014)

⋮

This is a **booming topic**, and many new applications frequently arise.

Fractional integrals

Euler Γ-function

The Γ **function** $\Gamma(z)$ is defined for complex numbers with a positive real part via the convergent improper integral:

$$\Gamma(z) = \int_0^{+\infty} x^{z-1} e^{-x} \, dx, \qquad \mathfrak{R}(z) > 0,$$

and then extended by **analyitic continuation** to a *meromorphic* function that is holomorphic in the whole complex plane except zero and the negative integers, where the function has simple poles.
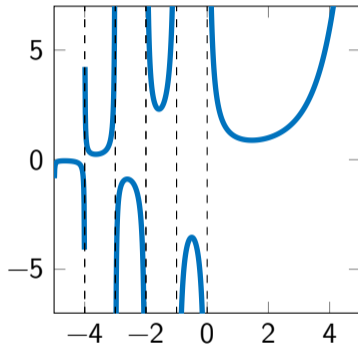
$$\Gamma(z+1) = z \, \Gamma(z)$$

Bounded in:

$$S = \{z \in \mathbb{C} : \mathfrak{R}z \in [1, 2)\}$$

A formula for repeated integration

Swapping Integrals

If $G(x, t)$ is jointly continuous on $[c, b] \times [c, b]$:

$$\int_c^x \mathrm{d}x_1 \int_c^{x_1} G(x_1, x_2) \mathrm{d}x_2 = \int_c^x \mathrm{d}x_2 \int_{x_2}^x G(x_1, x_2) \mathrm{d}x_1.$$

A formula for repeated integration

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Fubini's Theorem

Given $(X, \mathfrak{S}_X, \mu_x)$, $(Y, \mathfrak{S}_Y, \mu_y)$ measure spaces with σ-finite complete measures μ_x, μ_y on the σ-algebras $\mathfrak{S}_X$, and $\mathfrak{S}_Y$. If the function $f(x, y)$ is integrable on the product $X \times Y$ w.r.t. the product measure $\mu = \mu_x \times \mu_y$, then the following equality holds true

$$\int_{X \times Y} f(x, y)\,\mathrm{d}\mu = \int_Y \mathrm{d}\mu_y \int_X f(x, y)\mathrm{d}\mu_x.$$

A formula for repeated integration

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$$\int_{X \times Y} f(x, y) \, \mathrm{d}\mu = \int_Y \mathrm{d}\mu_y \int_X f(x, y)\mathrm{d}\mu_x.$$

A formula for repeated integration

The indefinite integral of order $n \in \mathbb{N}$ of function $f(t)$ is given by

$$I^n_{c,t} f(t) = \int_c^t \cdots \int_c^t f(t)\, \mathrm{d}t \cdots \mathrm{d}t = \frac{1}{(n-1)!} \int_c^t (t-\tau)^{n-1} f(\tau)\, \mathrm{d}\tau,$$

$$I^n_{t,c} f(t) = \int_t^c \cdots \int_t^c f(t)\, \mathrm{d}t \cdots \mathrm{d}t = \frac{1}{(n-1)!} \int_c^t (\tau-t)^{n-1} f(\tau)\, \mathrm{d}\tau.$$

- Can be proved **by induction** using Fubini's Theorem/the previous formula,

A formula for repeated integration

The indefinite integral of order $n \in \mathbb{N}$ of function $f(t)$ is given by

$$I_{c,t}^n f(t) = \int_c^t \cdots \int_c^t f(t)\,\mathrm{d}t \cdots \mathrm{d}t = \frac{1}{\Gamma(n)} \int_c^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau,$$

$$I_{t,c}^n f(t) = \int_t^c \cdots \int_t^c f(t)\,\mathrm{d}t \cdots \mathrm{d}t = \frac{1}{\Gamma(n)} \int_c^t (\tau-t)^{n-1} f(\tau)\,\mathrm{d}\tau.$$

- Can be proved **by induction** using Fubini's Theorem/the previous formula,
- We have introduced the Γ function so let's use it,

A formula for repeated integration

Cauchy's formula

The indefinite integral of order $n \in \mathbb{N}$ of function $f(t)$ is given by

$$I_{c,t}^n f(t) = \int_c^t \cdots \int_c^t f(t) \, \mathrm{d}t \cdots \mathrm{d}t = \frac{1}{\Gamma(n)} \int_c^t (t-\tau)^{n-1} f(\tau) \, \mathrm{d}\tau,$$

$$I_{t,c}^n f(t) = \int_t^c \cdots \int_t^c f(t) \, \mathrm{d}t \cdots \mathrm{d}t = \frac{1}{\Gamma(n)} \int_c^t (\tau-t)^{n-1} f(\tau) \, \mathrm{d}\tau.$$

- Can be proved **by induction** using Fubini's Theorem/the previous formula,
- We have introduced the Γ function so let's use it,
- Now we use it to move from the integer case to the **real one**.

Riemann–Liouville Fractional Integrals

Riemann–Liouville Fractional Integral

Let $\Re\alpha > 0$, and let $f \in \mathbb{L}^1([a,b])$. Then for $t \in [a,b]$ we call

$$I_{[a,t]}^{\alpha} f(t) = {_a}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \, \mathrm{d}\tau,$$

$$I_{[t,b]}^{\alpha} f(t) = {_a}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) \, \mathrm{d}\tau.$$

the **Riemann–Liouville** fractional integrals of f of order α, we set it to be the identity operator whenever $\alpha = 0$.

💡 **the idea** is that we have substituted the integer number n of repetition of the integral with the real order α,

❓ but does this makes sense?

RL Fractional Integrals: properties - I

Theorem (Existence).

Lef $f \in \mathbb{L}^1[a,b]$, and $\alpha > 0$. Then, the integral $I_{[a,t]}^\alpha f(t)$ exists for almost every $t \in [a,b]$. Moreover, the function $I_{[a,t]}^\alpha f$ itself is also an element of $\mathbb{L}^1[a,b]$.

Proof. It is sufficient to recognize that we can write the integral in question as a convolution on $\mathbb{R}$, indeed:

$$\int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau = \int_{-\infty}^{+\infty} \Phi_1(t-\tau) \Phi_2(\tau) \, \mathrm{d}\tau,$$

where

$$\Phi_1(u) = \begin{cases} u^{\alpha-1}, & \text{for } 0 < t \leq b-a, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } \Phi_2(u) = \begin{cases} f(u), & \text{for } u \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

By construction both the Φ_j, $j = 1,2$, are in $\mathbb{L}^1(\mathbb{R})$, and thus the integral exists and is a member of $\mathbb{L}^1$ as a convolution of $\mathbb{L}^1$ functions (We are using again *Fubini's Theorem*).

RL Fractional Integrals: properties - II

Theorem (Semigroup property).

The RL fractional integral operators $\{I_c^\alpha : \mathbb{L}^1[a, b] \to \mathbb{L}^1[a, b], \ \alpha \geq 0\}$ form a commutative semigroup with respect to the concatenation operation, that is

$$I_c^\alpha(I_c^\beta f(t)) = I_c^{\alpha+\beta} f(t)), \ \text{and} \ I_c^\beta(I_c^\alpha f(t)) = I_c^{\alpha+\beta} f(t)).$$

The neutral element of this semigroup is the I_c^0 operator.

Proof. We prove it for one side, the other is analogous.

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The neutral element of this semigroup is the I_c^0 operator.

Proof. We have just proved that the integral exists, then by using *Fubini's theorem* we can interchange the order of integration:

$$I_{[a,t]}^\alpha I_{[a,t]}^\beta f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_\tau^x (x-t)^{\alpha-1}(t-\tau)^{\beta-1} f(\tau) \, d\tau \, dt$$

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We now use the substitution $t = \tau + s(x - \tau)$, $\mathrm{d}t = (x-\tau)\mathrm{d}s$.

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$$I_{[a,t]}^\alpha I_{[a,t]}^\beta f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau) \int_0^1 [(x-\tau)(1-s)]^{\alpha-1}[s(x-\tau)]^{\beta-1}(x-\tau) \,\mathrm{d}s \,\mathrm{d}\tau$$

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$$I_{[a,t]}^\alpha I_{[a,t]}^\beta f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau)(x-\tau)^{\alpha+\beta-1} \int_0^1 (1-s)^{\alpha-1} s^{\beta-1} \, \mathrm{d}s \, \mathrm{d}\tau.$$

RL Fractional Integrals: properties - II

Euler's β-function

The Euler's β-function is defined as:

$$\beta(x,y) \triangleq \int_0^1 u^{x-1}(1-u)^{y-1}\, du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \Re x > 0, \Re y > 0,$$

Proof. We have just proved that the integral exists, then by using *Fubini's theorem* we can interchange the order of integration. We now use the substitution $t = \tau + s(x-\tau)$, $\mathrm{d}t = (x-\tau)\mathrm{d}s$. We obtain:

$$I_{[a,t]}^{\alpha} I_{[a,t]}^{\beta} f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau)(x-\tau)^{\alpha+\beta-1} \int_0^1 (1-s)^{\alpha-1} s^{\beta-1}\, \mathrm{d}s\, \mathrm{d}\tau.$$

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$$I_{[a,t]}^\alpha I_{[a,t]}^\beta f(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-\tau)^{\alpha+\beta-1} f(\tau)\,\mathrm{d}\tau = I_{[a,t]}^{\alpha+\beta} f(x), \quad \text{a.e. on } [a,b].$$

The same works also if we exchange α and β, while we have the 0th order operator being the neutral element by definition.

RL Fractional Integrals: properties - III

A note on regularity.

Observe that in the proof we could say something more on the regularity of the resulting functions. Indeed if f is a continuous function on $[a, b]$, then also $I_{[a,t]}^{\alpha} f$ is continuous.

Therefore we have that also the concatenation $I_{[a,t]}^{\alpha} I_{[a,t]}^{\beta}$ and $I_{[a,t]}^{\alpha+\beta}$ are continuous. Then what we have proved is that we have two continuous function that are **almost everywhere equal**, and therefore they most coincide everywhere. Furthermore, if $f \in \mathbb{L}^1[a, b]$ and $\alpha + \beta \geq 1$ we can use Semigroup property to write

$$I_{[a,t]}^{\alpha} I_{[a,t]}^{\beta} f = I_{[a,t]}^{\alpha+\beta} f = I_{[a,t]}^{\alpha+\beta-1} I_{[a,t]}^{1} f, \ a.e.$$

Now, since $I_{[a,t]}^{1} f$ is continuos, we also get that the other two way of writing it are continuous, and thus we can conclude the equality everywhere by the same argument as before.

Computing a Riemann–Liouville fractional integral.

$$I_{[0,t]}^{\alpha} t^{\mu} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\mu} \, d\tau,$$

This should be the simplest possible example, and indeed it is as simple as using again the **Euler β Function**:

$$\beta(x,y) \triangleq \int_0^1 u^{x-1}(1-u)^{y-1} \, du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \Re x > 0, \Re y > 0.$$
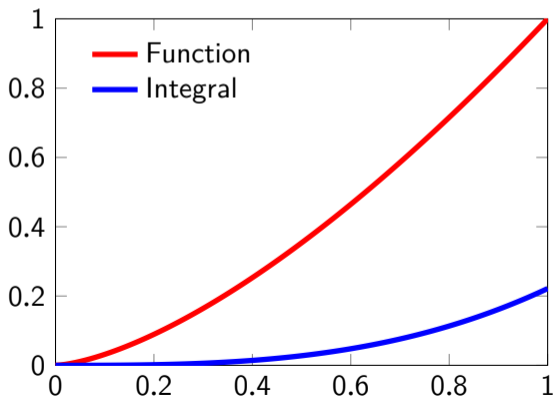
To obtain it, we do the substitution for $u = \frac{\tau}{t}$, then

$$\begin{aligned} I_{[0,t]}^{\alpha} t^{\mu} &= \frac{t^{\alpha+\mu}}{\Gamma(\alpha)} \int_0^1 u^{\mu}(1-u)^{\alpha-1} \, du \\ &= \frac{t^{\alpha+\mu}}{\Gamma(\alpha)} \frac{\Gamma(\mu+1)\Gamma(\alpha)}{\Gamma(\alpha+\mu+1)} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}. \end{aligned}$$

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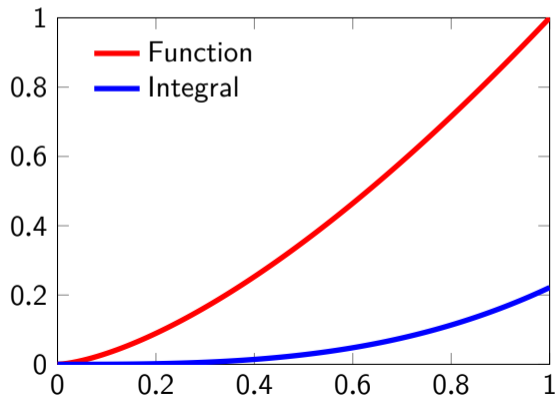
```
t = linspace(0,1,100);
I = @(alpha,mu,t)
↪   gamma(mu+1)*t.^(alpha+mu)/
↪   gamma(alpha+mu+1);
mu = 1.5;
alpha = 1.5;
plot(t,t.^mu,'r-',t,I(alpha,mu,t),
↪   'b-','Linewidth',2);
legend('Function','Integral');
```

Computing a Riemann–Liouville fractional integral.

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legend('Function','Integral');
```

😖 They are hard to compute!

Quadratures for Fractional Integrals

💡 Quadrature idea

Let us assume that $f(t)$ is suitably smooth on an interval (a, b). Let

$$h = \frac{b - a}{N}, \quad t_k = a + kh, \quad \text{with } k = 0, 1, 2, \ldots, N, \quad N \in \mathbb{N}$$

then we can approximate for $t = t_N$ the fractional integral as

$$\left. {}_aD_b^{-\alpha}f(t)\right|_{t=t_N} = \frac{1}{\Gamma(\alpha)} \int_a^{t_N} (t_N - \tau)^{\alpha-1} f(\tau) \, \mathrm{d}\tau = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_k - \tau)^{\alpha-1} f(\tau) \, \mathrm{d}\tau.$$

We approximate $f(x)$ with a polynomial $p(x)$ such that we can compute exactly the involved integrals, this yields quadratures by the usual look

$$\left. {}_aD_b^{-\alpha}f(t)\right|_{t=t_N} \approx \sum_{k=0}^{N-1} \omega_k f(t_k).$$

Piecewise constant approximation

We approximate $f(t)$ on the intervals $[t_k, t_k + 1)$, $k = 0, \ldots, N-1$, selecting

$$f(t) \approx p(t) \equiv p(t_k), \quad t \in [t_k, t_k + 1), \ k = 0, 1, \ldots, N-1,$$

from which we get the formula

$$\begin{aligned}
{}_aD_b^{-\alpha}f(t)\big|_{t=t_N} &\approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{N-1} f(t_k) \int_{t_k}^{t_k+1} (t_N - \tau)^{\alpha-1}\,\mathrm{d}\tau = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{N-1} f(t_k) \left[-\frac{1}{\alpha}(t_N - \tau)^{\alpha} \right]_{t_k}^{t_{k+1}} \\
&= \sum_{k=0}^{N-1} f(t_k) \frac{1}{\alpha\Gamma(\alpha)} \left[(t_N - t_k)^{\alpha} - (t_N - t_{k+1})^{\alpha} \right] \\
&= \sum_{k=0}^{N-1} f(t_k) \frac{1}{\alpha\Gamma(\alpha)} \left[(a + hn - a - kh)^{\alpha} - (a + hn - a - (k+1)h)^{\alpha} \right] \\
&= \sum_{k=0}^{N-1} f(t_k) \frac{h^{\alpha}}{\Gamma(\alpha+1)} \left[(n-k)^{\alpha} - (N-k-1)^{\alpha} \right] = \sum_{k=0}^{N-1} b_{N-k-1} f(t_k),
\end{aligned}$$

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from which we get the formula

$$_a D_b^{-\alpha} f(t)\big|_{t=t_N} \approx \sum_{k=0}^{N-1} b_{N-k-1} f(t_k),$$

where we have defined

$$b_k = \frac{h^\alpha}{\Gamma(\alpha + 1)} [(k+1)^\alpha - k^\alpha], \qquad 0 \leq k \leq N-1.$$

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Analogously we get the case in which we select the **right approximation**

$$f(t) \approx p(t) \equiv p(t_{k+1}), \quad t \in [t_k, t_k + 1), \ k = 0, 1, \ldots, N-1,$$

and, more generally, for the **weighted formula** in which we select

$$f(t) \approx p(t) \equiv \lambda p(t_k) + (1-\lambda)p(t_{k+1}), \quad t \in [t_k, t_k + 1), \ k = 0, 1, \ldots, N-1, \ \lambda \in [0,1].$$

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We approximate $f(t)$ on the intervals $[t_k, t_k + 1)$, $k = 0, \ldots, N - 1$, selecting

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from which we get the formula

$$_aD_b^{-\alpha}f(t)\big|_{t=t_N} \approx \sum_{k=0}^{N-1} b_{N-k-1}f(t_k), \qquad b_k = \frac{h^\alpha}{\Gamma(\alpha+1)}[(k+1)^\alpha - k^\alpha].$$

The general **weighted formula** is then given by

$$_aD_b^{-\alpha}f(t)\big|_{t=t_N} \approx \sum_{k=0}^{N-1} b_{N-k-1}\left[\lambda p(t_k) + (1-\lambda)p(t_{k+1})\right], \quad \lambda \in [0,1].$$

Implementation

This is a simple procedure to implement

```
function I = constfracint(f,a,t,alpha,N,lambda)
%CONSTFRACINT computes the fractional integral with the weighted piecewise
%constant approximation of the function f between a and t, over N uniformly
%distributed intervals.
h  = (t-a)/N;
tk = (a:h:t)';
b  = zeros(N,1);
for k=0:N-1
   b(k+1) = (k+1)^alpha - k^alpha;
end
b = h^alpha*b/gamma(alpha+1);
p = f(tk);
I = flipud(b)'*(lambda*p(1:N) + (1-lambda)*p(2:N+1));
end
```

Implementation - II

And we can **test the results** using the fractional integral we have computed by hand

```
f     = @(t,mu) t.^mu;
Itrue = @(alpha,mu,t) gamma(mu+1)*t.^(alpha+mu)/ gamma(alpha+mu+1);
mu    = 1;
alpha = 1.5;

N = 100;
lambda = 1;
I = constfracint(@(t) f(t,mu),0,1,alpha,N,1);
fprintf('Relative error is: %e\n',abs(I-Itrue(alpha,mu,1))./abs(Itrue(alpha,mu,1)));
```

That returns us
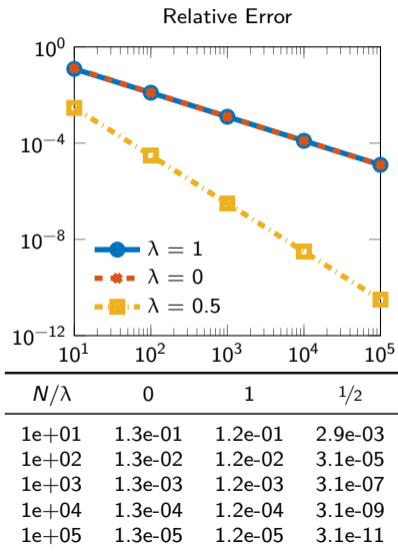
```
Relative error is: 1.246939e-02
```

But what about convergence?

Convergence

Fractional Newton-Cotes formula

Lef $f(t)$ be approximated by a polynomial $p_{k,r}(t)$ of degree r on the grid points $\{t_k = t_0^{(k)}, \ldots, t_r^{(k)} = t_{k+1}\}$. Then the error estimate for an $f \in \mathcal{C}^{r+1}([a,b])$ on each sub-interval $[t_k, t_{k+1}]$ is given by

$$f(t) - p_{k,r}(t) = \frac{f^{(r+1)}(\tau_k)}{(r+1)!} \prod_{j=0}^{r}(t - t_j^{(k)}),$$

for $r \in \mathbb{N}$, $t, \tau_k \in [t_k, t_{k+1}]$, i.e., the formula is of order $O(h^{r+1})$.

Relative Error

N/λ	0	1	1/2
1e+01	1.3e-01	1.2e-01	2.9e-03
1e+02	1.3e-02	1.2e-02	3.1e-05
1e+03	1.3e-03	1.2e-03	3.1e-07
1e+04	1.3e-04	1.2e-04	3.1e-09
1e+05	1.3e-05	1.2e-05	3.1e-11

Convergence

Proof. The interpolating polynomial can be expressed in the **Lagrange basis**

$$p_{k,r}(t) = \sum_{i=0}^{r} l_{k,i}(t) f(t_i^{(k)}), \quad l_{k,i}(t) = \prod_{\substack{j=0 \\ j \neq i}}^{r} \frac{t - t_j^{(k)}}{t_i^{(k)} - t_j^{(k)}}, \quad 0 \leq i \leq r, \ t \in [t_k, t_{k+1}].$$

Then the fractional Newton-Coates formula si given by

$$\left. _aD_b^{-\alpha} f(t) \right|_{t=t_N} \approx \left. _aD_b^{-\alpha} p_{k,r}(t) \right|_{t=t_N} = \sum_{k=0}^{N-1} \sum_{i=0}^{r} C_{i,N}^{(k)} f(t_i^{(k)}),$$

for

$$C_{i,N}^{(k)} = \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} (t_N - \tau)^{\alpha-1} l_{k,i}(\tau) \, \mathrm{d}\tau.$$

Convergence

Proof. Then the fractional Newton-Coates formula si given by

$$\left. {}_aD_b^{-\alpha}f(t)\right|_{t=t_N} \approx \left. {}_aD_b^{-\alpha}p_{k,r}(t)\right|_{t=t_N} = \sum_{k=0}^{N-1}\sum_{i=0}^{r} C_{i,N}^{(k)}f(t_i^{(k)}),$$

from which we obtain the error estimate as

$$
\begin{aligned}
\left| {}_aD_b^{-\alpha}f(t) - {}_aD_b^{-\alpha}p_{k,r}(t)\right| \leq & \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(t_N-\tau)^{\alpha-1}\left|f(\tau)-p_{k,r}(\tau)\right|\,\mathrm{d}\tau \\
\leq & \max_{t\in[a,t_N]}\frac{|f^{(r+1)}(t)|}{(r+1)!\Gamma(\alpha)}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(t_N-\tau)^{\alpha-1}\prod_{j=0}^{r}|\tau-t_j^{(k)}|\,\mathrm{d}\tau \\
\leq & \max_{t\in[a,t_N]}\left|f^{(r+1)}(t)\right|\frac{h^{r+1}}{(r+1)!\Gamma(\alpha)}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(t_N-\tau)^{\alpha-1}\,\mathrm{d}\tau.
\end{aligned}
$$

Convergence

Proof. Then the fractional Newton-Coates formula si given by

$$\left. {_aD_b^{-\alpha}}f(t)\right|_{t=t_N} \approx \left. {_aD_b^{-\alpha}}p_{k,r}(t)\right|_{t=t_N} = \sum_{k=0}^{N-1}\sum_{i=0}^{r} C_{i,N}^{(k)} f(t_i^{(k)}),$$

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\leq & \max_{t\in[a,t_N]}\frac{|f^{(r+1)}(t)|}{(r+1)!\Gamma(\alpha)}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(t_N-\tau)^{\alpha-1}\prod_{j=0}^{r}|\tau-t_j^{(k)}|\,\mathrm{d}\tau \\
\leq & \max_{t\in[a,t_N]}\left|f^{(r+1)}(t)\right|\frac{h^{r+1}}{(r+1)!\Gamma(\alpha+1)}(t_N-t_0)^{\alpha}.
\end{aligned}
$$

Convergence

Proof. Then the fractional Newton-Coates formula si given by

$$\left. {}_aD_b^{-\alpha}f(t)\right|_{t=t_N} \approx \left. {}_aD_b^{-\alpha}p_{k,r}(t)\right|_{t=t_N} = \sum_{k=0}^{N-1}\sum_{i=0}^{r} C_{i,N}^{(k)} f(t_i^{(k)}),$$

from which we obtain the error estimate as

$$\left| {}_aD_b^{-\alpha}f(t) - {}_aD_b^{-\alpha}p_{k,r}(t)\right| \in O(h^{r+1}).$$

Remark
The error estimate does not coincide completely with the classical one for Newton-Coates formulas, this is due to the nonsymmetry of the integral kernel $(t_N' - t)^{\alpha-1}$.

Suggested exercises, and some extensions

(i) Rewrite (and implement) the fractional weighted constant approximation for the *other-sided* Riemann-Liuoville fractional integral,

(ii) Denote with $t_{k+1/2} = {}^{t_k + t_{k+1}}/_2$ on each sub-interval $[t_k, t_{k+1}]$, approximate $f(t)$ with a *piecewise quadratic polynomial*, derive and implement the fractional Simpson's formula
   – ⚠ The closed form of the coefficients for this case is cumbersome…

Extensions

By mimicking the usual procedure for deriving collocation/spectral type quadrature formulas, we could approximate $f(t)$ by using, e.g., Jacobi polynomials to obtain the related quadrature formulas (when you have obtained formulas for Jacobi, then *Chebyshev* and *Legendre* follow with relative "ease").

Riemann–Liouville Fractional Derivatives

Now that we've gotten a little bit of familiarity with Riemann–Liouville integral operators, we can finally **introduce the corresponding differential operators**.

> ### 💡 The key idea
>
> Let f be a function having a continuous nth derivative on the interval $[a, b]$, and let $m \in \mathbb{N}$ be such that $m > n$, then
>
> $$\frac{d^n}{dt^n} f(t) = \frac{1}{(m-n-1)!} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-n-1} f(\tau) \, \mathrm{d}\tau = \frac{d^m}{dt^m} I_a^{m-n} f,$$
>
> simply by employing the Fundamental Theorem of (Classical) Calculus
>
> $$f = \frac{d^{m-n}}{dt^{m-n}} I_a^{m-n} f,$$
>
> and applying the operator $\frac{d^n}{dt^n}$ to both side of it.

Riemann–Liouville Fractional Derivatives

Now that we've gotten a little bit of familiarity with Riemann–Liouville integral operators, we can finally **introduce the corresponding differential operators**.

> 💡 The key idea now we go from integers to real numbers!
>
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>
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>
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> and applying the operator $\frac{d^n}{dt^n}$ to both side of it.

Riemann–Liouville Fractional Derivatives

Substitute the integer n with a real positive number α and select an $m \in \mathbb{N}$ s.t. $m > \alpha$.

RL Derivative

Let $\alpha \in \mathbb{R}_+$ and $m = \lceil \alpha \rceil$, we define the Riemann-Liouville operator $_{\text{RL}}D_a^\alpha$ as

$$_{\text{RL}}D_a^\alpha f(t) \triangleq \frac{d^m}{dt^m} I_a^{m-\alpha} f(t),$$

and we set $_{\text{RL}}D_a^0$ to the identity operator.

⚠ The right-hand side of our definition remains valid, **but** now the resulting operator depends on the choice of the point a.

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⚠️ The right-hand side of our definition remains valid, **but** now the resulting operator depends on the choice of the point a.

❓ for what functions f does this definition make sense?

Riemann–Liouville Fractional Derivatives Existence

The $\mathbb{A}^n$ functions

We call $\mathbb{A}^n[a, b]$, or simply $\mathbb{A}^n$ when the interval is clear from the context, the space of function with an **absolutely continuous** $(n-1)$st derivative, i.e., the functions f for which there exists almost everywhere a (generalized) nth derivative function $g \in \mathbb{L}^1[a, b]$ for which holds

$$f^{(n-1)}(t) = f^{(n-1)}(a) + \int_a^t g(\tau)\, \mathrm{d}\tau.$$

Remind: For a compact interval:

continuously differentiable $\subseteq$ Lipschitz continuous $\subseteq$ absolutely continuous $\subseteq$ bounded variation $\subseteq$ differentiable almost everywhere

Example: $f(t) = \sqrt[3]{t}$ is absolutely continuous on any bounded interval I **but** not Lipschitz continuous on any interval I such that $0 \in I$.

Riemann–Liouville Fractional Derivatives Existence

Theorem (Existence)

Lef $f \in \mathbb{A}^1[a, b]$, and $0 < \alpha < 1$. Then $_{RL}D_a^\alpha f(t)$ exists almost everywhere in $[a, b]$. Moreover, $_{RL}D_a^\alpha f(t) \in \mathbb{L}^p$ for $1 \leq p < \alpha^{-1}$ and

$$_{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^\alpha} + \int_a^t f'(\tau)(t-\tau) \, d\tau \right).$$

Proof. We use directly the two definitions

$$_{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(\tau)(t-\tau)^{-\alpha} \, d\tau$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left(f(a) + \int_a^\tau f'(s) \, ds \right) (t-\tau)^{-\alpha} \, d\tau$$

Riemann–Liouville Fractional Derivatives Existence

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Lef $f \in \mathbb{A}^1[a, b]$, and $0 < \alpha < 1$. Then $_{\text{RL}} D_a^\alpha f(t)$ exists almost everywhere in $[a, b]$. Moreover, $_{\text{RL}} D_a^\alpha f(t) \in \mathbb{L}^p$ for $1 \leq p < \alpha^{-1}$ and

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$$\begin{aligned}
_{\text{RL}} D_a^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left(f(a) + \int_a^\tau f'(s) \, \mathrm{d}s \right) (t-\tau)^{-\alpha} \, \mathrm{d}\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(f(a) \int_a^t \frac{\mathrm{d}t}{(x-t)^\alpha} + \int_a^t \int_a^\tau f'(s)(t-\tau)^{-\alpha} \, \mathrm{d}s \, \mathrm{d}\tau \right)
\end{aligned}$$

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$$_{\text{RL}}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^\alpha} + \int_a^t f'(\tau)(t-\tau)\,\mathrm{d}\tau \right).$$

Proof. We use directly the two definitions, and apply again *Fubini's Theorem*

$$_{\text{RL}}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(f(a) \int_a^t \frac{\mathrm{d}t}{(x-t)^\alpha} + \int_a^\tau \int_a^t f'(s)(t-\tau)^{-\alpha}\,\mathrm{d}s\,\mathrm{d}\tau \right)$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^\alpha} + \frac{d}{dt} \int_a^\tau \int_a^t f'(s)(t-\tau)^{-\alpha}\,\mathrm{d}s\,\mathrm{d}\tau \right)$$

$$\text{(Fubini)} = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^\alpha} + \frac{d}{dt} \int_a^t f'(s) \frac{(t-s)^{1-\alpha}}{1-\alpha}\,\mathrm{d}s \right),$$

Riemann–Liouville Fractional Derivatives Existence

Theorem (Existence)

Lef $f \in \mathbb{A}^1[a, b]$, and $0 < \alpha < 1$. Then $_{RL}D_a^\alpha f(t)$ exists almost everywhere in $[a, b]$. Moreover, $_{RL}D_a^\alpha f(t) \in \mathbb{L}^p$ for $1 \le p < \alpha^{-1}$ and

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Proof. We use directly the two definitions, and finally Leibniz rule for the derivative of integral functions,

$$_{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^\alpha} + \frac{d}{dt} \int_a^t f'(s) \frac{(t-s)^{1-\alpha}}{1-\alpha} \, \mathrm{d}s \right),$$

$$\text{(Leibniz)} = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^\alpha} + \int_a^t f'(\tau)(t-\tau) \, \mathrm{d}\tau \right).$$

Computing our first RL derivatives

To keep things simple we can compute, first of all, the fractional derivative of order $\alpha \in (0,1)$ of the **constant function** $f(t) = 1$ in $[0,t]$:

We simply apply the previous representation theorem, and thus:

$$_{\text{RL}}D_{[0,1]}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(0)}{(t-0)^{\alpha}} + \int_{0}^{t}f'(\tau)(t-\tau)\,\mathrm{d}\tau\right) =$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{1}{(t-0)^{\alpha}} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

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$$
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&= \frac{1}{\Gamma(1-\alpha)}\frac{1}{(t-0)^{\alpha}} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}
\end{aligned}
$$
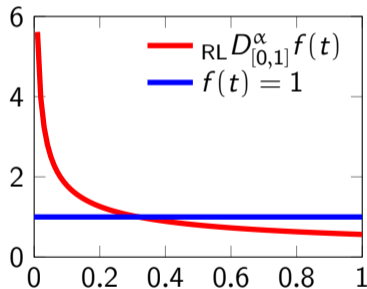
The RL derivative of a constant is not zero!

Computing our first RL derivatives

Let $f(t) = (t - a)^\beta$ for some $\beta > -1$ and compute its RL derivative of order $\alpha > 0$ on an interval $[a, b]$.

First we compute the **fractional integral** part of the definition:

$$
\begin{aligned}
I^\alpha_{[a,t]} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (\tau - a)^\beta (t - \tau)^{\alpha - 1} \, \mathrm{d}\tau = \\
&= \frac{1}{\Gamma(\alpha)} \int_0^{t-a} s^\beta (t - a - s)^{\alpha - 1} \, \mathrm{d}s = \leftarrow \left(\int_0^x s^{\beta - 1} (x - s)^{\alpha - 1} \mathrm{d}s = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} x^{\alpha + \beta - 1} \right) \\
&= \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} (t - a)^{\alpha + \beta},
\end{aligned}
$$

Computing our first RL derivatives

Let $f(t) = (t-a)^\beta$ for some $\beta > -1$ and compute its RL derivative of order $\alpha > 0$ on an interval $[a, b]$.

First we compute the **fractional integral** part of the definition:

$$I_{[a,t]}^\alpha f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta},$$

Then we just have to compute the derivative with the correct indexes

$$_{\mathrm{RL}}D_{[0,1]}^\alpha f(t) = \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} I_{[a,t]}^{\lceil\alpha\rceil-\alpha} f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\lceil\alpha\rceil-\alpha+\beta+1)} \left. \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}(\cdot-a)^{\lceil\alpha\rceil-\alpha+\beta}\right|_t,$$

now, if $\alpha - \beta \in \mathbb{N}$ the right-hand side vanishes ($\lceil\alpha\rceil$-derivative of a polynomial of lower degree), if $\alpha - \beta \notin \mathbb{N}$, we find

$$_{\mathrm{RL}}D_{[0,1]}^\alpha f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha}.$$

Summary and anticipations

We did

- ✅ Definition and properties of Riemann–Liouville Integrals,
- ✅ Some examples of Fractional Newton-Cotes formulas for RL integral computations,
- ✅ Definition and existence of Riemann–Liouville Derivatives,
- ✅ A couple of by-hand computations of RL derivatives of simple functions.

Next up

- 📋 Properties and interactions between Riemann–Liouville Integrals and Derivatives,
- 📋 The Caputo fractional derivative,
- 📋 An introduction to Fractional Differential Equations.

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