An introduction to fractional calculus

Fundamental ideas and numerics

Fabio Durastante
Università di Pisa
fabio.durastante@unipi.it
fdurastante.github.io

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RL Fractional Integrals and Derivatives

Riemann–Liouville Fractional Integral

Let $\Re\alpha > 0$, and let $f \in \mathbb{L}^1([a,b])$. Then for $t \in [a,b]$ we define

$$I^{\alpha}_{[a,t]} f(t) = {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau,$$

$$I^{\alpha}_{[t,b]} f(t) = {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau.$$

Riemann–Liouville Fractional Derivative

Let $\Re\alpha > 0$, $m = \lceil \alpha \rceil$, and $f \in \mathbb{A}^m([a,b])$, Then for $t \in [a,b]$ we define

$${}_{RL}D^{\alpha}_{[a,t]} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau)\,\mathrm{d}\tau,$$

$${}_{RL}D^{\alpha}_{[t,b]} f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_t^b (\tau-t)^{m-\alpha-1} f(\tau)\,\mathrm{d}\tau.$$

RL Derivatives Properties

RL integrals have a semigroup property, d/dt has it, so what about RL Derivatives?

Theorem

Assume that $\alpha_1, \alpha_2 \geq 0$. Moreover let $\phi \in \mathbb{L}^1([a,b])$, and $f = I^{\alpha_1+\alpha_2}_{[a,b]}\phi$. Then,

$$ {}_{RL}D^{\alpha_1}_{[a,t]}\,{}_{RL}D^{\alpha_2}_{[a,t]} f = {}_{RL}D^{\alpha_1+\alpha_2}_{[a,t]}. $$

Proof. We use the definition and the assumption on f,

$$ {}_{RL}D^{\alpha_1}_{[a,t]}\,{}_{RL}D^{\alpha_2}_{[a,t]} f = {}_{RL}D^{\alpha_1}_{[a,t]}\,{}_{RL}D^{\alpha_2}_{[a,t]} I^{\alpha_1+\alpha_2}_{[a,b]}\phi = \frac{d^{\lceil\alpha_1\rceil}}{dt^{\lceil\alpha_1\rceil}} I^{\lceil\alpha_1\rceil-\alpha_1}_{[a,b]} \frac{d^{\lceil\alpha_2\rceil}}{dt^{\lceil\alpha_2\rceil}} I^{\lceil\alpha_2\rceil-\alpha_2}_{[a,b]} I^{\alpha_1+\alpha_2}_{[a,b]}\phi $$

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$$ {}_{RL}D_{[a,t]}^{\alpha_1}\, {}_{RL}D_{[a,t]}^{\alpha_2} f = {}_{RL}D_{[a,t]}^{\alpha_1+\alpha_2}. $$

Proof. We use the definition and the assumption on f, then we use the *semigroup property* for integrals

$$ \begin{aligned} {}_{RL}D_{[a,t]}^{\alpha_1}\, {}_{RL}D_{[a,t]}^{\alpha_2} f =& {}_{RL}D_{[a,t]}^{\alpha_1}\, {}_{RL}D_{[a,t]}^{\alpha_2} I_{[a,b]}^{\alpha_1+\alpha_2}\phi = \frac{d^{\lceil\alpha_1\rceil}}{dt^{\lceil\alpha_1\rceil}} I_{[a,b]}^{\lceil\alpha_1\rceil-\alpha_1} \frac{d^{\lceil\alpha_2\rceil}}{dt^{\lceil\alpha_2\rceil}} I_{[a,b]}^{\lceil\alpha_2\rceil-\alpha_2} I_{[a,b]}^{\alpha_1+\alpha_2}\phi \\ =& \frac{d^{\lceil\alpha_1\rceil}}{dt^{\lceil\alpha_1\rceil}} I_{[a,b]}^{\lceil\alpha_1\rceil-\alpha_1} \frac{d^{\lceil\alpha_2\rceil}}{dt^{\lceil\alpha_2\rceil}} I_{[a,b]}^{\lceil\alpha_2\rceil+\alpha_1}\phi = \frac{d^{\lceil\alpha_1\rceil}}{dt^{\lceil\alpha_1\rceil}} I_{[a,b]}^{\lceil\alpha_1\rceil-\alpha_1} \frac{d^{\lceil\alpha_2\rceil}}{dt^{\lceil\alpha_2\rceil}} I_{[a,b]}^{\lceil\alpha_2\rceil} I_{[a,b]}^{\alpha_1}\phi \end{aligned} $$

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Proof. We use the definition and the assumption on f, then we use the *semigroup property* for integrals, and since orders of the integral and differential operators involved are in $\mathbb{N}$

$$ \begin{aligned} {}_{RL}D^{\alpha_1}_{[a,t]}\,{}_{RL}D^{\alpha_2}_{[a,t]} f =& \frac{d^{\lceil \alpha_1 \rceil}}{dt^{\lceil \alpha_1 \rceil}} I^{\lceil \alpha_1 \rceil - \alpha_1}_{[a,b]} \frac{d^{\lceil \alpha_2 \rceil}}{dt^{\lceil \alpha_2 \rceil}} I^{\lceil \alpha_2 \rceil}_{[a,b]} I^{\alpha_1}_{[a,b]}\phi = \frac{d^{\lceil \alpha_1 \rceil}}{dt^{\lceil \alpha_1 \rceil}} I^{\lceil \alpha_1 \rceil - \alpha_1}_{[a,b]} I^{\alpha_1}_{[a,b]}\phi = \frac{d^{\lceil \alpha_1 \rceil}}{dt^{\lceil \alpha_1 \rceil}} I^{\lceil \alpha_1 \rceil}_{[a,b]}\phi \\ =& \phi. \end{aligned} $$

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RL integrals have a semigroup property, $^{d}/_{dt}$ has it, so what about RL Derivatives?

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Assume that $\alpha_1, \alpha_2 \geq 0$. Moreover let $\phi \in \mathbb{L}^1([a,b])$, and $f = I^{\alpha_1+\alpha_2}_{[a,b]}\phi$. Then,

$$_{RL}D^{\alpha_1}_{[a,t]}\,_{RL}D^{\alpha_2}_{[a,t]} f = \,_{RL}D^{\alpha_1+\alpha_2}_{[a,t]}.$$

Proof. We use the definition and the assumption on f, then we use the *semigroup property* for integrals, and since orders of the integral and differential operators involved are in $\mathbb{N}$. This way we proved that: $_{RL}D^{\alpha_1}_{[a,t]}\,_{RL}D^{\alpha_2}_{[a,t]} f = \phi$. Now we work on the other part, that is analogous:

$$_{RL}D^{\alpha_1+\alpha_2}_{[a,t]} f = \frac{d^{\lceil \alpha_1+\alpha_2 \rceil}}{dt^{\lceil \alpha_1+\alpha_2 \rceil}} I^{\lceil \alpha_1+\alpha_2 \rceil - \alpha_1 - \alpha_2}_{[a,b]} f = \frac{d^{\lceil \alpha_1+\alpha_2 \rceil}}{dt^{\lceil \alpha_1+\alpha_2 \rceil}} I^{\lceil \alpha_1+\alpha_2 \rceil}_{[a,b]} I^{-\alpha_1-\alpha_2}_{[a,b]} I^{\alpha_1+\alpha_2}_{[a,b]} \phi = \phi.$$

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Assume that $\alpha_1, \alpha_2 \geq 0$. Moreover let $\phi \in \mathbb{L}^1([a,b])$, and $f = I_{[a,b]}^{\alpha_1+\alpha_2}\phi$. Then,

$$ {}_{RL}D_{[a,t]}^{\alpha_1}\, {}_{RL}D_{[a,t]}^{\alpha_2} f = {}_{RL}D_{[a,t]}^{\alpha_1+\alpha_2}. $$

An observation on the hypothesis

The crucial hypothesis for the proof has been having $f = I_{[a,b]}^{\alpha_1+\alpha_2}\phi$. This is **not technical**, consider $f(t) = \sqrt{t}$, and $\alpha_1 = \alpha_2 = 1/2$, then we have computed in the last lecture

$$ {}_{RL}D_{[0,t]}^{1/2}\sqrt{t} = 0, \Rightarrow \; {}_{RL}D_{[0,t]}^{1/2}\, {}_{RL}D_{[0,t]}^{1/2}\sqrt{t} = 0, $$

but ${}_{RL}D_{[0,t]}^{1} = \frac{d}{dt}\sqrt{t} = 1/2\sqrt{t} \neq 0$. The condition on f implies both the needed regularity, and regulates how $f(t) \to 0$ as $t \to a$. **Other example.** Consider the same function with $\alpha_1 = 1/2$, $\alpha_2 = 3/2$.

RL Derivatives Properties - II

Theorem

Let $\alpha \geq 0$. Then, for every $f \in \mathbb{L}^1([a,b])$

$$ {}_{RL}D^{\alpha}_{[a,t]} I^{\alpha}_{[a,t]} f = f \qquad \text{a.e.} $$

Proof. The case $\alpha = 0$ descend from the definitions, both operators are the identity. For $\alpha > 0$, let $m = \lceil \alpha \rceil$, then we use the definition of ${}_{RL}D^{\alpha}_{[a,t]}$ and the semigroup property of fractional integration

$$ {}_{RL}D^{\alpha}_{[a,t]} I^{\alpha}_{[a,t]} f = \frac{d^m}{dt^m} I^{m-\alpha}_{[a,t]} I^{\alpha}_{[a,t]} f = \frac{d^m}{dt^m} I^{m}_{[a,t]} f = f(t). $$

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Theorem

Let $\alpha \geq 0$. Then, for every $f \in \mathbb{L}^1([a,b])$

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Thus we have proved that the RL derivative is a **left inverse** of the RL integral, unfortunately we cannot claim that it is the right inverse.

Theorem

Let $\alpha > 0$. If there exists some $\phi \in \mathbb{L}^1([a,b])$ such that $f = I^{\alpha}_{[a,t]}\phi$ then

$$ I^{\alpha}_{[a,t]}\, {}_{RL}D^{\alpha}_{[a,t]} f = f. $$

Proof. This is an immediate consequence of the left-inverse property, since

$$ I^{\alpha}_{[a,t]}\, {}_{RL}D^{\alpha}_{[a,t]} f = I^{\alpha}_{[a,t]}\, {}_{RL}D^{\alpha}_{[a,t]} I^{\alpha}_{[a,t]} \phi = I^{\alpha}_{[a,t]} \phi = f. $$

RL Derivatives Properties - II

Theorem

Let $\alpha \geq 0$. Then, for every $f \in \mathbb{L}^1([a,b])$

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Let $\alpha > 0$. If there exists some $\phi \in \mathbb{L}^1([a,b])$ such that $f = I^{\alpha}_{[a,t]}\phi$ then

$$ I^{\alpha}_{[a,t]}\, {}_{RL}D^{\alpha}_{[a,t]} f = f. $$

What happens in the general case?

RL Derivatives Properties - III

Theorem

Let $\alpha > 0$, and $m = \lfloor \alpha \rfloor + 1$. Assume that f is such that $I^{m-\alpha}_{[a,t]} f \in \mathbb{A}^m([a,b])$. Then,

$$I^{\alpha}_{[a,t]} \, {}_{RL}D^{\alpha}_{[a,t]} f = f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \lim_{z \to a^+} \frac{d^{m-k-1}}{dz} I^{m-\alpha}_{[a,z]} f(z).$$

That reduces to

$$I^{\alpha}_{[a,t]} \, {}_{RL}D^{\alpha}_{[a,t]} f = f(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \lim_{z \to a^+} I^{1-\alpha}_{[a,z]} f(z), \text{ for } 0 < \alpha < 1.$$

- As for the semigroup property this is an issue of regularity and of going rapidly enough to zero at the beginning of the interval,
- The analogous property can be written also for the *other-sided* RL derivatives.

RL - Combinations, products and compositions

Linear combination descend easily from the definition.

Theorem

Let $f_1, f_2 : [a,b] \to \mathbb{R}$ such that ${}_{RL}D^{\alpha}_{[a,t]} f_1$, and ${}_{RL}D^{\alpha}_{[a,t]} f_1$ exist almost everywhere. Then, for $c_1, c_2 \in \mathbb{R}$ we have ${}_{RL}D^{\alpha}_{[a,t]}(c_1 f_1 + c_2 f_2)$ exists almost everywhere, and

$${}_{RL}D^{\alpha}_{[a,t]}(c_1 f_1 + c_2 f_2) = c_1 \, {}_{RL}D^{\alpha}_{[a,t]} f_1 + c_2 \, {}_{RL}D^{\alpha}_{[a,t]} f_2.$$

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Leibniz' formula for Riemann–Liouville operators, doesn't come so easily

Theorem (Leibniz' formula for Riemann–Liouville operators)

Let $\alpha > 0$, and assume f and g analytic on $(a - h, a + h)$ for some $h > 0$. Then,

$${}_{RL}D^{\alpha}_{[a,t]}[fg](t) = \sum_{k=0}^{\lfloor \alpha \rfloor} \binom{\alpha}{k} {}_{RL}D^{k}_{[a,t]} f(t) \, {}_{RL}D^{\alpha-k}_{[a,t]} g(t) + \sum_{k=\lfloor \alpha \rfloor + 1}^{+\infty} \binom{\alpha}{k} {}_{RL}D^{k}_{[a,t]} f(t) \, I^{k-\alpha}_{[a,t]} g(t),$$

for $t \in (a, a + h/2)$.

RL - Combinations, products and compositions - II

For compositions we need to recall first a result for integer-order derivatives

Francesco da Paola Virginio Secondo Maria Faà di Bruno's Lemma

If g and f are functions with a sufficient number of derivatives and $n \in \mathbb{N}$, then

$$\frac{d^n}{dt^n}[g(f(\cdot))](t) = \sum \left(\frac{d^k}{dt^k} g \right) (f(t)) \prod_{\mu=1}^{n} \left(\frac{d^\mu}{dt^\mu} f(t) \right)^{b_\mu},$$

where the sum is over all partitions of $\{1, 2, \ldots, n\}$, and for each partition k is its number of blocks and b_j is the number of blocks with exactly j elements.

For a proof (and the history) see (Johnson 2002).

RL - Combinations, products and compositions - II

For compositions we need to recall first a result for integer-order derivatives, then we can look at its extension

Faà di Bruno's formula for RL operators

If f and g are *regular enough* we have

$$
{}_{RL}D^{\alpha}_{[a,t]}[fg](t) = \sum_{k=1}^{+\infty} \binom{\alpha}{k} \frac{k!(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{\ell=1}^{k} \left({}_{RL}D^{\ell}_{[a,t]} f\right)(g(t))
$$

$$
\sum_{(a_1,\cdots,a_k)\in A_{k,\ell}} \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{\frac{d^r}{dt^r} g(t)}{r}\right)^{a_r} + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} f(g(t)),
$$

where $(a_1,\ldots,a_k) \in A_{k,\ell}$ means that

$$
a_1,\ldots,a_k \in \mathbb{N}_0,\ \sum_{r=1}^{k} r a_r = k \text{ and } \sum_{r=1}^{k} a_r = \ell.
$$

What now?

We have put together **all the analogues of the instruments of classical calculus**, but what do we do with them now?

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What we would like to solve is:

$$ {}_{RL}D^{\alpha}_{[0,t]}\mathbf{y}(t) = f(t, \mathbf{y}(t)), \qquad \mathbf{y} : [0, T] \to \mathbb{R}^d,\ f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d. $$

Nevertheless, we have a problem! What we would like to solve is a Cauchy problem, so we need to put **initial conditions**, but last time we observed that

$$ {}_{RL}D^{\alpha}_{[0,t]}\mathbf{c} \neq 0, \qquad \mathbf{c} \in \mathbb{R}^d. $$

Therefore, we should equip the system with the **following initial conditions** instead

$$ {}_{RL}D^{\alpha-k}_{[0,t]}\mathbf{y}(0) = \mathbf{b}_k, \quad k = 1, 2, \ldots, \lceil\alpha\rceil - 1,\ \lim_{z\to 0^+} I^{\lceil\alpha\rceil-\alpha}_{[0,t]}\mathbf{y}(z) = b_{\lceil\alpha\rceil}. $$

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We could develop a theory for this, but **these conditions are physically difficult** to use, we don't get this type of initial data from the applications.

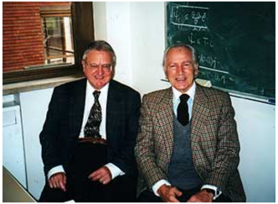
Caputo fractional derivatives

Caputo fractional derivative (Caputo 2008)

Let $\alpha \geq 0$, and $m = \lceil \alpha \rceil$. Then, we define the operator

$$ {}_C D^{\alpha}_{[a,t]} f = I^{m-\alpha}_{[a,t]} \frac{d^m}{dt^m} f, $$

whenever $\frac{d^m}{dt^m} f \in \mathbb{L}^1([a, b])$.



(R. Gorenflo, M. Caputo, Bologna 2000, source: fracalmo.org)

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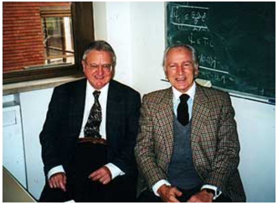

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- We have exchanged the order of the derivative and fractional integral operators.

"Chi cerca trova, chi ricerca ritrova." - E. De Giorgi

The concept occurred a certain number of times: (Džrbašjan and Nersesjan 1968; Gerasimov 1948; Gross 1947; Liouville 1832; Rabotnov et al. 1969).

So, what is the difference?

First of all, we have the result we wanted on constants $c \in \mathbb{R}$:

$$_C D^{\alpha}_{[a,t]} c = I^{m-\alpha}_{[a,t]} \frac{d^m}{dt^m} c = I^{m-\alpha}_{[a,t]} 0 = 0.$$

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We can put in relation the two operators with the following result

Theorem

Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. Moreover, assume that $f \in \mathbb{A}^m([a,b])$. Then,

$$_C D^{\alpha}_{[a,t]} f = {}_{RL} D^{\alpha}_{[a,t]} \left[f - T_{m-1}[f; a] \right] \text{ a.e. on } [a, b],$$

for $T_{m-1}[f; a]$ the Taylor polynomial of degree $m-1$ for the function f centered at a, with $T_{-1}[f; a] = 0$.

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$$\begin{aligned} {}_{RL}D^{\alpha}_{[a,t]} \left[f - T_{m-1}[f;a]\right] =& \frac{d^m}{dt^m} I^{m-\alpha}_{[a,t]} \left[f - T_{m-1}[f;a]\right] \\ =& \frac{d^m}{dt^m} \int_a^t \frac{(t-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} \left(f(\tau) - T_{m-1}[f;a](\tau)\right) \, \mathrm{d}\tau, \end{aligned}$$

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&= \frac{d^m}{dt^m} \int_a^t \frac{(t-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} \left(f(\tau) - T_{m-1}[f;a](\tau)\right) \mathrm{d}\tau,
\end{aligned}
$$

We apply a **partial integration**

$$
\begin{aligned}
* = &-\frac{1}{\Gamma(m-\alpha+1)} \left[(f(\tau) - T_{m-1}[f,a](\tau))(t-\tau)^{m-\alpha} \right]\Big|_{\tau=a}^{\tau=t} \\
&+ \frac{1}{\Gamma(m-\alpha+1)} \int_a^t (f'(\tau) - (T_{m-1}[f,a](\tau))')(t-\tau)^{m-\alpha} \mathrm{d}\tau.
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The terms in red are zero, and only the integral terms remain.

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\end{aligned}
$$

We apply a **partial integration** m times since $f \in \mathbb{A}^m([a,b])$:

$$
I^{m-\alpha}_{[a,t]}\left[f - T_{m-1}[f;a]\right] = I^{2m-\alpha}_{[a,t]} \frac{d^m}{dt^m}\left[f - T_{m-1}[f;a]\right] = I^{m}_{[a,t]} I^{m-\alpha}_{[a,t]} \frac{d^m}{dt^m}\left[f - T_{m-1}[f;a]\right],
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We apply a **partial integration** m times since $f \in \mathbb{A}^m([a,b])$:

$$
I^{m-\alpha}_{[a,t]}\left[f - T_{m-1}[f;a]\right] = I^{2m-\alpha}_{[a,t]} \frac{d^m}{dt^m}\left[f - T_{m-1}[f;a]\right] = I^{m}_{[a,t]} I^{m-\alpha}_{[a,t]} \frac{d^m}{dt^m}\left[f - T_{m-1}[f;a]\right],
$$

the mth derivative of the Taylor polynomial is zero (degree $m-1$).

So, what is the difference?

Proof. In the case $\alpha \in \mathbb{N}$ the result follows easily, since both quantities reduces to the integer order αth derivative. Therefore, we consider the case $\alpha \notin \mathbb{N}$ and $m = \lceil \alpha \rceil > \alpha$

$$
\begin{aligned}
{}_{RL}D^{\alpha}_{[a,t]}\left[f - T_{m-1}[f;a]\right] =& \frac{d^m}{dt^m} I^{m-\alpha}_{[a,t]}\left[f - T_{m-1}[f;a]\right] \\
=& \frac{d^m}{dt^m} \int_a^t \frac{(t-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} \left(f(\tau) - T_{m-1}[f;a](\tau)\right) \mathrm{d}\tau,
\end{aligned}
$$

We apply a **partial integration** m times since $f \in \mathbb{A}^m([a,b])$ and obtain the expression

$$
I^{m-\alpha}_{[a,t]}\left[f - T_{m-1}[f;a]\right] = I^{m}_{[a,t]} I^{m-\alpha}_{[a,t]} \frac{d^m}{dt^m} f.
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{}_{RL}D^{\alpha}_{[a,t]}\left[f - T_{m-1}[f;a]\right] &= \frac{d^m}{dt^m} I^{m-\alpha}_{[a,t]}\left[f - T_{m-1}[f;a]\right] \\
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$$

We reapply the mth derivative to the simplified expression:

$$
{}_{RL}D^{\alpha}_{[a,t]}\left[f - T_{m-1}[f;a]\right] = \frac{d^m}{dt^m} I^{m}_{[a,t]} I^{m-\alpha}_{[a,t]} \frac{d^m}{dt^m} f = I^{m-\alpha}_{[a,t]} \frac{d^m}{dt^m} f = {}_{C}D^{\alpha}_{[a,t]} f.
$$

An example of computation

Let $f(t) = (t-a)^\beta$ for some $\beta \geq 0$, then

$$
{}_{CA}D^{\alpha}_{[a,t]} f(t) = \begin{cases} 0, & \beta \in \{0,1,2,\ldots,\lceil\alpha\rceil - 1\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha}, & (\beta \in \mathbb{N} \wedge \beta \geq \lceil\alpha\rceil) \\ & \vee (\beta \notin \mathbb{N} \wedge \beta > \lceil\alpha\rceil - 1). \end{cases}
$$

Let us compare it with the Riemann-Liouville case:

$$
{}_{\mathrm{RL}}D^{\alpha}_{[0,1]} f(t) = \begin{cases} 0, & \alpha - \beta \in \mathbb{N}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha}, & \alpha - \beta \notin \mathbb{N}. \end{cases}
$$

! The two operators have different kernels and domain.

Caputo fractional derivatives - Properties

We can rewrite all the properties we have seen for RL derivatives for the Caputo version.

Theorem. (Caputo Derivatives Properties)

Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$

(i) ${}_C D^{\alpha}_{[a,t]} f = {}_{RL} D^{\alpha}_{[a,t]} f - \sum_{k=0}^{m-1} {}^{f^{(k)}(a)}/_{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}$,

(ii) ${}_C D^{\alpha}_{[a,t]} f = {}_{RL} D^{\alpha}_{[a,t]} f$ iff f has a zero of order m at a,

(iii) If f is continuous, ${}_C D^{\alpha}_{[a,t]} I^{\alpha}_{[a,t]} f = f$,

(iv) If $f \in \mathbb{A}^m([a,b])$ then $I^{\alpha}_{[a,t]} {}_C D^{\alpha}_{[a,t]} = f(t) - \sum_{k=0}^{m-1} {}^{f^{(k)}(a)}/_{k!} (x-a)^k$,

(v) If $f \in \mathcal{C}^k([a,b])$, $\alpha, \beta > 0$ s.t. $\exists \ell \in \mathbb{N}$ $\ell \leq k$ and $\alpha, \alpha+\beta \in [\ell-1, \ell]$ then ${}_C D^{\alpha}_{[a,t]} {}_C D^{\beta}_{[a,t]} f = {}_C D^{\alpha+\beta}_{[a,t]} f$.

(vi) $f \in \mathcal{C}^{\mu}([a,b])$, $\alpha \in [0, \mu]$, then ${}_{RL} D^{\mu-\alpha}_{[a,t]} {}_C D^{\alpha}_{[a,t]} f = f^{(\mu)}$.

Caputo fractional derivatives - Properties

Theorem. (Caputo Derivatives Properties)

(vii) For $f_1, f_2 : [a,b] \to \mathbb{R}$, $c_1, c_2 \in \mathbb{R}$ then

$$ {}_C D^{\alpha}_{[a,t]}(c_1 f_1 + c_2 f_2) = c_1 \, {}_C D^{\alpha}_{[a,t]} f_1 + c_2 \, {}_C D^{\alpha}_{[a,t]} f_2 \text{ a.e. on } [a,b], $$

if ${}_C D^{\alpha}_{[a,t]} f_1$, ${}_C D^{\alpha}_{[a,t]} f_2$ exist a.e. on $[a,b]$,

(viii) (Leibniz) let $\alpha \in (0,1)$, f, g analytic on $(a-h, a+h)$, then

$$ {}_C D^{\alpha}_{[a,t]}[fg](t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} g(a)(f(t) - f(a)) + \left({}_C D^{\alpha}_{[a,t]}(g(t)\right) f(t) + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(I^{k-\alpha}_{[a,t]} g(t) \right) {}_C D^{k}_{[a,t]} f(t). $$

They can all be proved by mimicking the proofs for the RL derivative.

Fractional ODEs with Caputo Derivatives

Let's restart with the differential equation, but now written in terms of Caputo Derivatives

$$\alpha > 0, \quad m = \lceil \alpha \rceil, \qquad \begin{cases} {}_{C}D^{\alpha}_{[0,t]}\mathbf{y}(t) = f(t, \mathbf{y}(t)), & t \in [0, T], \\ \dfrac{d^k \mathbf{y}(0)}{dt^k} = \mathbf{y}_0^{(k)}, & k = 0, 1, \ldots, m-1. \end{cases} \tag{FODE}$$

And we are now faced with the usual questions

- Is there any solution?
- If there is at least one, then how many there are?
- When it is all said and proved, how can we approximate it?

Fractional ODEs with Caputo Derivatives

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And we are now faced with the usual questions

- Is there any solution? $\longrightarrow$ This lecture
- If there is at least one, then how many there are?$\longrightarrow$ This lecture
- When it is all said and proved, how can we approximate it?$\rightsquigarrow$ The next one

A Peano existence theorem for first order equations

Theorem (Diethelm and Ford 2002, Theorem 2.1, 2.2)

Let $0 < \alpha$ and $m = \lceil \alpha \rceil$. Moreover let $\{y_0^{(k)} \in \mathbb{R}\}_{k=0}^{m-1}$, $K > 0$, and $h^* > 0$. We define

$$G = \left\{ (t, y) \,:\, t \in [0, h^*] \,:\, \left| y - \sum_{k=0}^{m-1} t^k y_0^{(k)} / k! \right| \leq K \right\},$$

and let the function $f : G \to \mathbb{R}$ be continuous. Furthermore, define

$$M = \sup_{(t,z)\in G} |f(t, z)|, \;\; h = \begin{cases} h^*, & \text{if } M = 0, \\ \min\{h^*, (K\Gamma(\alpha+1)/M)^{1/n}\}, & \text{else.} \end{cases}$$

Then, there exists a function $y \in \mathcal{C}([0, h])$ solving (FODE).

A Peano existence theorem for first order equations

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Let $0 < \alpha$ and $m = \lceil \alpha \rceil$. Moreover let $\{y_0^{(k)} \in \mathbb{R}\}_{k=0}^{m-1}$, $K > 0$, and $h^* > 0$. We define

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Then, there exists a function $y \in \mathcal{C}([0,h])$ solving (FODE).

To prove it we need a Lemma…and a bit of work.

A Peano existence theorem for first order equations

Lemma

Under the same hypotheses of the previous Theorem. A function $y \in \mathcal{C}([0,h])$ is a solution of the initial value problem (FODE) *if and only if* it is a solution of the nonlinear Volterra integral equation of the second kind

$$y(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\, \mathrm{d}\tau, \quad m = \lceil \alpha \rceil.$$

Proof. We need to prove both the implications.

A Peano existence theorem for first order equations

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Proof. We need to prove both the implications.($\Rightarrow$) First of all we have $y(t)$ being a continuous solution of the nonlinear Volterra equation. We apply on both side the Caputo derivative of order α

$${}_C D^{\alpha}_{[0,t]} y(t) = \underbrace{{}_C D^{\alpha}_{[0,t]} \left[\sum_{k=0}^{m-1} \frac{t^k}{k!} y_0^{(k)} \right]}_{=0 \ \lceil \alpha \rceil > m-1} + {}_C D^{\alpha}_{[0,t]} \left[\int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\,\mathrm{d}\tau \right],$$

A Peano existence theorem for first order equations

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$${}_C D^{\alpha}_{[0,t]} y(t) = {}_C D^{\alpha}_{[0,t]} I^{\alpha}_{[0,t]} f(t, y(t)) = f(t, y(t)), \; f \text{ is continuous.}$$

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Proof. We need to prove both the implications.($\Rightarrow$) follows by direct computation. ($\Leftarrow$) Is a bit more laborious. Let us define $z(t) = f(t, y(t)) \in \mathcal{C}[0,h]$, we can rewrite (FODE) as:

$$z(t) = f(t,y(t)) = {}_{C}D^{\alpha}_{[0,t]} y(t) = {}_{RL}D^{\alpha}_{[0,t]}(y - T_{m-1}[y;0](t)) = \frac{d^m}{dt^m} I_0^{m-\alpha}(y - T_{m-1}[y;0])(t),$$

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$$I^m_{[0,t]} z(t) = I^m_{[0,t]} \frac{d^m}{dt^m} I_0^{m-\alpha}(y - T_{m-1}[y;0])(t) = I_0^{m-\alpha}(y - T_{m-1}[y;0])(t) + q(t),$$

for a polynomial $q(t) \in \mathbb{P}_{\leq m-1}[t]$.

A Peano existence theorem for first order equations

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Under the same hypotheses of the previous Theorem. A function $y \in \mathcal{C}([0,h])$ is a solution of the initial value problem (FODE) *if and only if* it is a solution of the nonlinear Volterra integral equation of the second kind

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$I^m_{[0,t]} z(t) =$

$I_0^{m-\alpha}(y - T_{m-1}[y;0])(t) + q(t) \Rightarrow$

- $I^m_{[0,t]} z(t) = 0$ (at least) $m$times for $t = 0$,
- $y - T_{m-1}[y;0] = 0$ (at least) $m$times for $t = 0$,
- $I_0^{m-\alpha}(y - T_{m-1}[y;0])(t) = 0$ (at least) $m$times for $t = 0$,

A Peano existence theorem for first order equations

Lemma

Under the same hypotheses of the previous Theorem. A function $y \in \mathcal{C}([0,h])$ is a solution of the initial value problem (FODE) *if and only if* it is a solution of the nonlinear Volterra integral equation of the second kind

$$y(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\,\mathrm{d}\tau, \quad m = \lceil \alpha \rceil.$$

Proof. We need to prove both the implications. $(\Rightarrow)$ follows by direct computation. $(\Leftarrow)$ Is a bit more laborious. Let us define $z(t) = f(t, y(t)) \in \mathcal{C}[0,h]$, we can rewrite (FODE) as:

$I_{[0,t]}^{m} z(t) =$
$I_0^{m-\alpha}(y - T_{m-1}[y;0])(t) + q(t)$

Therefore $q(t) = 0$ (at least) $m$times for $t = 0$, but $\deg(q) \leq m-1 \Rightarrow q \equiv 0$.

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$$I_{[0,t]}^{m} z(t) = I_0^{m-\alpha}(y - T_{m-1}[y;0])(t)$$

and apply ${}_{RL}D_{[0,t]}^{m-\alpha}$ to both side of the equation.

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$$(y - T_{m-1}[y;0])(t) =_{RL} D^{m-\alpha}_{[0,t]} I^{m}_{[0,t]} z(t) = \frac{d}{dt} I^{1+\alpha-m}_{[0,t]} I^{m}_{[0,t]} z(t) = \frac{d}{dt} I^{1+\alpha}_{[0,t]} z(t) = I^{\alpha}_{0} z(t).$$

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Under the same hypotheses of the previous Theorem. A function $y \in \mathcal{C}([0,h])$ is a solution of the initial value problem (FODE) *if and only if* it is a solution of the nonlinear Volterra integral equation of the second kind

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by recalling the definitions of $T_{m-1}[y,0]$ and the RL-integral.

A Peano existence theorem for first order equations

The other two results we will need (and that we are not going to prove) are

Theorem (Ascoli-Arzelà)

Lef $F \subset \mathcal{C}([a,b])$ for some $a < b$, and assume the sets to be equipped with the supremum norm. Then F is *relatively compact*[1] in $\mathcal{C}([a,b])$ if F is

- *uniformly bounded*, $\exists\, C > 0$ s.t. $\|f\|_\infty \leq C \;\forall\, f \in F$,
- *equicontinuous* $\forall\, \varepsilon > 0\; \exists\, \delta > 0$ such that $\forall\, f \in F$ and all $x, x* \in [a,b]$ with $|x - x^*| < \delta$ we have $|f(x) - f(x^*)| < \varepsilon$.

Schauder's Fixed Point Theorem

Lef (E, d) be a complete metric space, let U be a closed convex subset of E, and let $A : U \to U$ be a mapping such that the set $\{Au \;:\; u \in U\}$ is *relatively compact*[1] in E. Then A has at least one fixed point.

[1] A subset whose closure is compact.

A Peano existence theorem for first order equations

Let us look again at the statement of the Theorem.

Theorem (Diethelm and Ford 2002, Theorem 2.1, 2.2)

Let $0 < \alpha$ and $m = \lceil \alpha \rceil$. Moreover let $\{y_0^{(k)} \in \mathbb{R}\}_{k=0}^{m-1}$, $K > 0$, and $h^* > 0$. We define

$$G = \left\{ (t, y) \,:\, t \in [0, h^*] \,:\, \left| y - \sum_{k=0}^{m-1} t^k y_0^{(k)}/k! \right| \leq K \right\},$$

and let the function $f : G \to \mathbb{R}$ be continuous. Furthermore, define

$$M = \sup_{(t,z)\in G} |f(t, z)|, \; h = \begin{cases} h^*, & \text{if } M = 0, \\ \min\{h^*, (K^{\Gamma(\alpha+1)}/M^{1/n}\}, & \text{else.} \end{cases}$$

Then, there exists a function $y \in \mathcal{C}([0, h])$ solving (FODE).

A Peano existence theorem for first order equations

Proof. If $M = 0$, then $f(x, y) = 0$ for all $(x, y) \in G$, then we can explicitly write the solution as

$$y : [0, h] \to \mathbb{R} \qquad y(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_0^{(k)},$$

therefore a solution exists.

A Peano existence theorem for first order equations

Proof. If $M > 0$, let us apply the **Lemma** and rewrite our problem as a Volterra equation:

$$y(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\,\mathrm{d}\tau, \quad m = \lceil \alpha \rceil,$$

and introduce the polynomial T satisfying the boundary condition and the space U

$$T(t) = \sum_{k=0}^{m-1} \frac{x^k}{k!} y_0^{(k)}, \quad U = \{y \in \mathcal{C}([0,h]) : \|y - T\|_\infty \leq K\}.$$

A Peano existence theorem for first order equations

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- U is closed and convex,
- $U \subset \mathcal{C}([0,h])$,

$\Rightarrow$ U is a non empty Banach space (at least $T \in U$).

A Peano existence theorem for first order equations

Proof. If $M > 0$, let us apply the **Lemma** and rewrite our problem as a Volterra equation:

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and introduce the polynomial T satisfying the boundary condition and the space U

$$T(t) = \sum_{k=0}^{m-1} \frac{x^k}{k!} y_0^{(k)}, \quad U = \{y \in \mathcal{C}([0,h]) \,:\, \|y - T\|_\infty \leq K\}.$$

Let us define the operator:

$$(Ay)(t) = T(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\,\mathrm{d}\tau.$$

A Peano existence theorem for first order equations

Proof. If $M > 0$, let us apply the **Lemma** and rewrite our problem as a Volterra equation:

$$y = Ay, \quad (Ay)(t) = T(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\,\mathrm{d}\tau.$$

we have to prove that A has a fixed point by the following steps:

1. proving that $Ay \in U$,
2. showing that $A(U) = \{Au \,:\, u \in U\}$ is *relatively compact* (Ascoli-Arzelà),
3. apply Schauder's Fixed Point Theorem for the victory.

A Peano existence theorem for first order equations

Proof. *Step 1.* Let us take $0 \le t_1 \le t_2 \le h$

$$\begin{aligned}
|(Ay)(t_1) - (Ay)(t_2)| =& \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1 - \tau)^{\alpha-1} f(\tau, y(\tau))\,\mathrm{d}\tau - \int_0^{t_2} (t_2 - \tau)^{\alpha-1} f(\tau, y(\tau))\,\mathrm{d}\tau \right| \\
=& \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \left[(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right] f(\tau, y(\tau))\,\mathrm{d}\tau \right. \\
& \left. - \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} f(\tau, y(\tau))\,\mathrm{d}\tau \right| \\
\le& \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} \left| (t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right| \,\mathrm{d}\tau + \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1}\,\mathrm{d}\tau \right)
\end{aligned}$$

A Peano existence theorem for first order equations

Proof. *Step 1.* Let us take $0 \leq t_1 \leq t_2 \leq h$

$$
\begin{aligned}
|(Ay)(t_1) - (Ay)(t_2)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1 - \tau)^{\alpha-1} f(\tau, y(\tau))\, \mathrm{d}\tau - \int_0^{t_2} (t_2 - \tau)^{\alpha-1} f(\tau, y(\tau))\, \mathrm{d}\tau \right| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \left[(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right] f(\tau, y(\tau))\, \mathrm{d}\tau \right. \\
&\qquad \left. - \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} f(\tau, y(\tau))\, \mathrm{d}\tau \right| \\
&\leq \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} \left| (t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right| \mathrm{d}\tau + \frac{(t_2 - t_1)^{\alpha}}{\alpha} \right)
\end{aligned}
$$

A Peano existence theorem for first order equations

Proof. *Step 1.* Let us take $0 \le t_1 \le t_2 \le h$

$$|(Ay)(t_1) - (Ay)(t_2)| \le \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} \left| (t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right| \, \mathrm{d}\tau + \frac{(t_2 - t_1)^{\alpha}}{\alpha} \right).$$

If $\alpha = 1$ the first integral vanishes.
If $\alpha < 1$, $\alpha - 1 < 0$, and hence $(t_1 - \tau)^{\alpha-1} \ge (t_2 - \tau)^{\alpha-1}$, thus we remove the $|\cdot|$ and

$$\int_0^{t_1} \left| (t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right| = \frac{1}{\alpha}(t_1^{\alpha} - t_2^{\alpha} + (t_2 - t_1)^{\alpha}) \le \frac{1}{\alpha}(t_2 - t_1)^{\alpha}.$$

If $\alpha > 1$ we have $(t_1 - \tau)^{\alpha-1} \le (t_2 - \tau)^{\alpha-1}$

$$\int_0^{t_1} \left| (t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right| = \frac{1}{\alpha}(t_2^{\alpha} - t_1^{\alpha} - (t_2 - t_1)^{\alpha}) \le \frac{1}{\alpha}(t_2^{\alpha} - t_1^{\alpha}).$$

A Peano existence theorem for first order equations

Proof. *Step 1.* Let us take $0 \leq t_1 \leq t_2 \leq h$

$$|(Ay)(t_1) - (Ay)(t_2)| \leq \begin{cases} {}^{2M}\!/_{\Gamma(\alpha+1)}(t_2 - t_1)^{\alpha}, & \alpha \leq 1, \\ {}^{M}\!/_{\Gamma(\alpha+1)}\left((t_2 - t_1)^{\alpha} + t_2^{\alpha} - t_1^{\alpha}\right), & \alpha > 1. \end{cases}$$

Therefore,

A Peano existence theorem for first order equations

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Therefore,

- Ay is continuous since $|(Ay)(t_1) - (Ay)(t_2)| \to 0$ for $t_2 \to t_1$,

A Peano existence theorem for first order equations

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Therefore,

- Ay is continuous since $|(Ay)(t_1) - (Ay)(t_2)| \longrightarrow 0$ for $t_2 \longrightarrow t_1$,
- for $y \in U$ and $t \in [0, h]$ we find

$$|(Ay)(t) - T(t)| = \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) \right| \le \frac{1}{\Gamma(\alpha+1)} M t^\alpha \le \frac{1}{\Gamma(\alpha+1)} M h^\alpha$$

$$\left(\text{Hp: } h < K \frac{\Gamma(\alpha+1)}{M}^{1/n} \right) \le \frac{1}{\Gamma(\alpha+1)} M \frac{K\Gamma(\alpha+1)}{M} = K.$$

A Peano existence theorem for first order equations

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Therefore,

- Ay is continuous since $|(Ay)(t_1) - (Ay)(t_2)| \to 0$ for $t_2 \to t_1$,
- for $y \in U$ and $t \in [0, h]$ we find $|(Ay)(t) - T(t)| \le K$

$\Rightarrow$ $Ay \in U$ if $y \in U$.

A Peano existence theorem for first order equations

Proof. Our plan:

- ✔ proving that $Ay \in U$,
2. showing that $A(U) = \{Au \,:\, u \in U\}$ is *relatively compact* (Ascoli-Arzelà),
3. apply Schauder's Fixed Point Theorem for the victory.

Step 2. First we prove that the set is *bounded*, let $z \in A(U)$ and $t \in [0, h]$

$$\begin{aligned}|z(t)| = |(Ay)(t)| &\leq \|T\|_\infty + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau, y(\tau))| \,\mathrm{d}\tau \\ &\leq \|T\|_\infty + \frac{1}{\Gamma(\alpha+1)} M h^\alpha \leq \|T\|_\infty + K.\end{aligned}$$

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$$|z(t)| \leq \|T\|_\infty + K.$$

For the *emicontinuity*, let $0 \leq t_1 \leq t_2 \leq h$ we found (for $\alpha \leq 1$)

$$|(Ay)(t_1) - (Ay)(t_2)| \leq \frac{2M}{\Gamma(\alpha+1)}(t_2 - t_1)^\alpha \leq \frac{2M}{\Gamma(\alpha+1)}\delta^\alpha, \qquad \text{if } |t_2 - t_1| < \delta.$$

the expression on the right is independent of y, t_1, and t_2.

A Peano existence theorem for first order equations

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$$|z(t)| \leq \|T\|_\infty + K.$$

For the *emicontinuity*, let $0 \leq t_1 \leq t_2 \leq h$ we found (for $\alpha > 1$)

$$\begin{aligned}
|(Ay)(t_1) - (Ay)(t_2)| \leq & \frac{M}{\Gamma(\alpha+1)}((t_2 - t_1)^\alpha + t_2^\alpha - t_1^\alpha), \\
\text{(Mean Value Theorem)} \quad = & \frac{M}{\Gamma(\alpha+1)}((t_2 - t_1)^\alpha + \alpha(t_2 - t_1)\tau^{\alpha-1}), \quad \tau \in [t_1, t_2] \subseteq [0, h]
\end{aligned}$$

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$$\begin{aligned}|(Ay)(t_1) - (Ay)(t_2)| \leq & \frac{M}{\Gamma(\alpha+1)}((t_2 - t_1)^\alpha + t_2^\alpha - t_1^\alpha), \\ \leq & \frac{M}{\Gamma(\alpha+1)}((t_2 - t_1)^\alpha + \alpha(t_2 - t_1)h^{\alpha-1})\end{aligned}$$

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$$|z(t)| \leq \|T\|_\infty + K.$$

For the *emicontinuity*, let $0 \leq t_1 \leq t_2 \leq h$ we found (for $\alpha > 1$)

$$|(Ay)(t_1) - (Ay)(t_2)| \leq \frac{M}{\Gamma(\alpha+1)}(\delta^\alpha + \alpha\delta h^\alpha - 1), \quad \text{if } |t_2 - t_1| < \delta,$$

the expression on the right is again independent of y, t_1, and t_2.

A Peano existence theorem for first order equations

Proof. Our plan:

- ✔ proving that $Ay \in U$,
- ✔ showing that $A(U) = \{Au \, : \, u \in U\}$ is *relatively compact* (Ascoli-Arzelà),

3. apply Schauder's Fixed Point Theorem for the victory ✌.

Finally we have all the ingredients:

- $E = \mathcal{C}([0,h])$, $U = \{y \in \mathcal{C}([0,h]) \, : \, \|y - T\|_\infty \leq K\}$ is a closed, convex subset of E.
- We have proved that the operator A is such that $\{Au \, : \, u \in U\}$ is relatively compact in E.

$\Rightarrow$ By Schauder's Fixed Point Theorem we have the existence of *at least* a solution. $\square$

A Peano existence theorem for first order equations

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At last...

We have proved existence: what about uniqueness?

A programming idea

We could use the fixed-point iteration as an algorithm for obtaining a solution.

Uniqueness of the solution à-la-Picard-Lindelöf

As for the classical calculus case, to prove *uniqueness* we need Lipschitzianity of the system dynamics w.r.t. to the second component.

Uniqueness of the solution à-la-Picard-Lindelöf

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Weissinger's Fixed Point Theorem

Assume (U, d) to be a nonempty complete metric space, and let $\beta_j \geq 0$ for every $j \in \mathbb{N}_0$ and such that $\sum_{j=0}^{\infty} \beta_j$ converges. Furthermore, let the mapping $A : U \to U$ satisfy the inequality

$$d(A^j u, A^j v) \leq \beta_j d(u, v), \quad \forall j \in \mathbb{N}, \quad \forall\, u, v \in U.$$

Then A has a *uniquely determined fixed point* u^*. Moreover, for any $u_0 \in U$, the sequence $(A^j u_0)_{j=1}^{\infty}$ converge to this fixed point.

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Then A has a *uniquely determined fixed point u^**. Moreover, for any $u_0 \in U$, the sequence $(A^j u_0)_{j=1}^{\infty}$ converge to this fixed point.

The plan

- ♻ Reuse the same set U, and map A from the existence proof,
- ≤ Prove the inequality and give an expression of the α_j in term of the Lipschitz constant.

Uniqueness of the solution à-la-Picard-Lindelöf

Theorem

Let $0 < \alpha$ and $m = \lceil \alpha \rceil$. Moreover, let $y_0^{(0)}, \ldots, y_0^{(m-1)} \in \mathbb{R}$, $K > 0$, and $h^* > 0$. We define the same set G:

$$G = \left\{ (t, y) \, : \, t \in [0, h^*] \, : \, \left| y - \sum_{k=0}^{m-1} t^k y_0^{(k)} / k! \right| \leq K \right\},$$

and let the function $f : G \to \mathbb{R}$ be continuous and Lipschitz w.r.t. the second entry

$$|f(t, y_1) - f(t, y_2)| \leq L |y_1 - y_2|,$$

for some $L > 0$ independently of t, y_1, and y_2. Then, for h such that

$$M = \sup_{(t,z) \in G} |f(t, z)|, \;\; h = \begin{cases} h^*, & \text{if } M = 0, \\ \min\{h^*, (K\Gamma(\alpha+1)/M)^{1/n}\}, & \text{else.} \end{cases}$$

there exist a uniquely defined $y \in \mathcal{C}[0, h]$ solving (FODE).

Uniqueness of the solution à-la-Picard-Lindelöf

Proof. We are under the same hypotheses of the **Existence Theorem**, thus (FODE) has a solution.
We prove **by induction** on j that

$$\|A^j y - A^j \tilde{y}\|_\infty \leq \frac{(Lt^\alpha)^j}{\Gamma(1+\alpha j)} \|y - \tilde{y}\|_\infty.$$

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Base case: $j = 0$ follows by the definition.

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Inductive hypothesis: we assume it true for $j-1$ and prove it for j.
Inductive step:

$$\begin{aligned}
\|A^j y - A^j \tilde{y}\|_\infty =& \|A(A^j - 1y) - A(A^{j-1}\tilde{y})\|_\infty \\
=& \frac{1}{\Gamma(\alpha)} \sup_{0\leq w\leq t} \left| \int_0^w (w-\tau)^{\alpha-1} \left[f(\tau, A^{j-1}y(\tau)) - f(\tau, A^{j-1}\tilde{y}(\tau)) \right] \,\mathrm{d}\tau \right| \\
\text{(Lipschitz)} \quad \leq& \frac{L}{\Gamma(\alpha)} \sup_{0\leq w\leq t} \int_0^w (w-\tau)^{\alpha-1} \left| A^{j-1}y(\tau) - A^{j-1}\tilde{y}(\tau) \right| \,\mathrm{d}\tau
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\leq& \frac{L}{\Gamma(\alpha)} \int_0^w (w-\tau)^{\alpha-1} \sup_{0\leq w\leq \tau} \left|A^{j-1}y(w) - A^{j-1}\tilde{y}(w)\right| \,\mathrm{d}\tau
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\text{(I.H.)} \quad \leq& \frac{L^j}{\Gamma(\alpha)\Gamma(1+\alpha(j-1))} \int_0^t (t-\tau)^{\alpha-1} t^{\alpha(j-1)} \sup_{0\leq w\leq\tau} |y(w) - \tilde{y}(w)| \,\mathrm{d}\tau
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\leq& \frac{L^j}{\Gamma(\alpha)\Gamma(1+\alpha(j-1))} \|y - \tilde{y}\|_\infty \frac{\Gamma(\alpha)\Gamma(1+\alpha(j-1))}{\Gamma(1+\alpha j)} t^{\alpha j}
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We proved **by induction** on j that

$$\|A^j y - A^j \tilde{y}\|_\infty \leq \frac{(Lt^\alpha)^j}{\Gamma(1+\alpha j)} \|y - \tilde{y}\|_\infty = \alpha_j \|y - \tilde{y}\|_\infty, \qquad \alpha_j = \frac{(Lh)^\alpha}{\Gamma(1+\alpha j)}.$$

To apply Weissinger's Fixed Point Theorem we need to prove that the series $\displaystyle\sum_{j=0}^{+\infty} \alpha_j = \sum_{j=0}^{+\infty} \frac{(Lh)^\alpha}{\Gamma(1+\alpha j)}$ converges.

Mittag-Leffler

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^\alpha}{\Gamma(\alpha k+1)}, \quad \alpha > 0 \qquad \text{is an entire function.}$$

State of the art

We have proved that the *Cauchy problem*

$$\alpha > 0, \quad m = \lceil \alpha \rceil, \qquad \begin{cases} {}_C D^{\alpha}_{[0,t]} \mathbf{y}(t) = f(t, \mathbf{y}(t)), & t \in [0, T], \\ \dfrac{d^k \mathbf{y}(0)}{dt^k} = \mathbf{y}_0^{(k)}, & k = 0, 1, \ldots, m-1. \end{cases}$$

admits

- for f continuous a *local* solution in $\mathcal{C}([0, h])$, $h < h^*$,
- for f continuous and Lipschitz in the second entry a *local* and *unique* solution in $\mathcal{C}([0, h])$, $h < h^*$.

State of the art

We have proved that the *Cauchy problem*

$$\alpha > 0, \quad m = \lceil \alpha \rceil, \qquad \begin{cases} {}_C D^{\alpha}_{[0,t]} \mathbf{y}(t) = f(t, \mathbf{y}(t)), & t \in [0, T], \\ \dfrac{d^k \mathbf{y}(0)}{dt^k} = \mathbf{y}_0^{(k)}, & k = 0, 1, \ldots, m-1. \end{cases}$$

admits

- for f continuous a *local* solution in $\mathcal{C}([0, h])$, $h < h^*$,
- for f continuous and Lipschitz in the second entry a *local* and *unique* solution in $\mathcal{C}([0, h])$, $h < h^*$.

For classical ODEs this is the point in which one starts proving *extension results* for the solutions. They exist also for the Fractional case. We are going to state them without proof.

Extension results

Corollary

Assume the hypotheses of the existence Theorem, but substitute G with the domain of definition of f, i.e., $G = [0, h^*] \times \mathbb{R}$. Moreover, assume that f is continuous and that there exist constants $c_1 \geq 0, c_2 \geq 0$, $0 \leq \mu < 1$ such that

$$f(t, y) \leq c_1 + c_2 |y|^{\mu}, \quad \forall (t, y) \in G.$$

Then, there exists a function $y \in \mathcal{C}([0, h^*])$ solving (FODE).

- Since G is no longer compact we need to demand a suitable bound explicitly, Weierstrasse Theorem no longer applies,
- A sufficient condition on f to imply the decay we need is for f to be continuous and bounded on G,

⚠ Our condition is violated already by linear equations!

Extension results

Theorem

Let $0 < \alpha$ and $m = \lceil \alpha \rceil$. Moreover, let $y_0^{(0)}, \dots, y_0^{(m-1)} \in \mathbb{R}$ and $h^* > 0$. We define the set $G = [0, h^*] \times \mathbb{R}$ and let $f : G \to \mathbb{R}$ be continuous and fulfill a Lipschitz condition with respect to the second variable with a Lipischitz constant L that is independent of t, y_1, and y_2. Then there exist a uniquely defined function $y \in \mathcal{C}([0, h^*])$ solving the (FODE).

- For a **proof** see the proof of Theorem 6.8 from (Diethelm 2010, pp 96-102) that is inspired by the proof for Volterra integral equations in (Linz 1985, Theorem 4.8).
- We can now solve linear equations

$$ {}_{CA}D^{\alpha}_{[0,t]} y(t) = f(t)y(t) + g(t), \qquad f, g \in \mathcal{C}([0, h^*]), \quad L = \|f\|_\infty < \infty. $$

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- Do we know hot to solve by hand any simple FODE?

Simple cases and representation formulas

The simplest ODE we know ho to solve is the *relaxation equation*

$$\mathbb{R} \ni \lambda < 0, \quad \begin{cases} y'(t) = \lambda y(t), & t \in [0, T], \\ y(0) = y_0, \end{cases} \qquad y(t) = y_0 \exp(\lambda t).$$

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Relaxation FODE

Let $\alpha > 0$, $m = \lceil \alpha \rceil$ and $\lambda \in \mathbb{R}$. The solution of the Cauchy problem

$$_{CA}D_{[0,t]} y(t) = \lambda y(t), \quad y(0) = y_0, \quad y^{(k)}(0) = 0, \quad k = 1, 2, \ldots, m-1,$$

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- The previous existence result tells us that the problem has indeed a *unique solution*.

Solution of the relaxation FODE

Two parameters Mittag-Leffler

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^{\alpha}}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0 \qquad \text{is an entire function.}$$

To see that this is the case we can use **Stirling formula** and **root test**

Stirling: $\Gamma(x+1) = ({}^{x}/_{e})^{x}\sqrt{2\pi x}(1 + o(1))$ for $x \to +\infty$,

Root test: $\sum_{n=1}^{+\infty} a_n$ converge absolutely if $C = \limsup_{n\to+\infty} \sqrt[n]{|a_n|} < 1$.

We write

$$a_j^{1/j} = \left(\frac{e}{j\alpha + \beta}\right)^{\alpha+\beta/j} (2\pi(\alpha j + \beta))^{-1/2j}(1 + o(1)) \to 0 \text{ for } j \to \infty.$$

Thus the radius of convergence is infinite.

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1. $y(0) = y_0 E_\alpha(0) = y_0$ since

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2. If $\alpha > 1,\ m \geq 2,\ y^{(k)}(0) = 0, \quad k = 1, 2, \dots, m-1$
$$y(t) = 1 + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots,$$
imposing the condition on the derivatives implies
$$y^{(k)}(t) = \frac{\lambda t^{\alpha-k}}{\Gamma(\alpha+1-k)} + \frac{\lambda^2 t^{2\alpha-k}}{\Gamma(2\alpha+1-k)} + \dots, \quad k = 1, 2, \dots, m-1.$$

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The existence theorems we have seen give us a solution $y(t) \in \mathcal{C}([0, h])$, can we have more?

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Regularity for ODEs

$$k \in \mathbb{N},\ f \in \mathcal{C}^{k-1}([y_0 - K, y_0 + k] \times \mathbb{R}),\ \begin{cases} y'(t) = f(t, y(t)), \\ y(0) = y_0 \end{cases} \Rightarrow\ y(t) \in \mathcal{C}^k.$$

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We can reuse our example computation:

$$f(t) = (t-a)^\beta,\; \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha},\; \beta \notin \mathbb{N} \wedge \beta > \lceil \alpha \rceil - 1$$

If we select $a = 0$, $\alpha = 1/2$, $\beta = 1/2$, then

$$\begin{cases} {}_{CA}D_{[0,t]} y(t) = \Gamma(3/2), \\ y(0) = 0, \end{cases} \Rightarrow\; y(t) = \sqrt{x}.$$

From an analytic right-hand side we got a non differentiable solution.

Why only continuous solutions?

Take-home message

Regularity of the right-hand side of the (FODE) is not sufficient to ensure regularity of the solution.

- Some more restrictive conditions under which regularity can be ensured can be found in (Diethelm 2007), to give an idea, one have to further ensure conditions for the zeros of $z(t) = f(t, y(t))$.
- Furthermore, if the solution of (FODE) is analytic, but not a polynomial of degree $\lceil \alpha \rceil - 1$, then f is not analytic.

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- Furthermore, if the solution of (FODE) is analytic, but not a polynomial of degree $\lceil \alpha \rceil - 1$, then f is not analytic.
- This will be important when we try do design *numerical methods*, since many results on convergence order usually rely on the regularity of the solution. Going high-order in the fractional settings is not in general an easy task!

The Mittag-Leffler Function

The $E_{\alpha,\beta}(z)$ takes the role of the exponential function when moving from ODEs to FODEs.

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^{\alpha}}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0.$$

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Laplace Transform

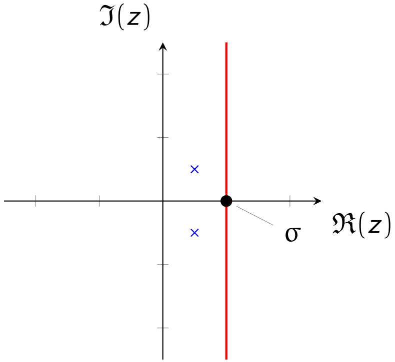
For a real- or complex-valued function $f(t)$ of the real variable t defined on $\mathbb{R}$ the (two-sided) Laplace transform is defined as

$$F(s) = \mathcal{L}\{f\}(s) = \int_{-\infty}^{+\infty} e^{-st} f(t)\,\mathrm{d}t.$$

Inverting the Laplace Transform

If we want fo compute $f(t)$ and have access to $F(s) = \mathcal{L}\{f\}(s)$ we can perform a *numerical inversion*, that is

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + \infty} e^{st} F(s) \, \mathrm{d}s.$$



where

- $(\sigma - i\infty, \sigma + i\infty)$ is called the Bromwich line,
- σ is such that all the singularities of $F(s)$ lies to the left $\mathfrak{R}(s) = \sigma$.

Branch lines

If $F(s)$ is a *multivalued function* we need to add a branch-cut to make the integrand single-valued.

Inverting the Laplace Transform

To numerically approximate the integral

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} F(s)\,\mathrm{d}s.$$

we **always need a change of variable**, the exponential term *oscillates wildly* and *decays slowly* along the Bromwich line.

We have to change the countour of integration to something more suitable, i.e., we change

$$s = s(u) \;\mapsto\; f(t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{s(u)t} F(s(u)) s'(u)\,\mathrm{d}u,$$

and then approximate the integral with the *trapezoidal rule* with spacing h

$$f_{h,N}(t) = \frac{h}{2\pi i} \sum_{k=-N}^{N} e^{s(u_k)t} F(s(u_k)) s'(u_k), \quad u_k = kh.$$

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Inverting the Laplace Transform

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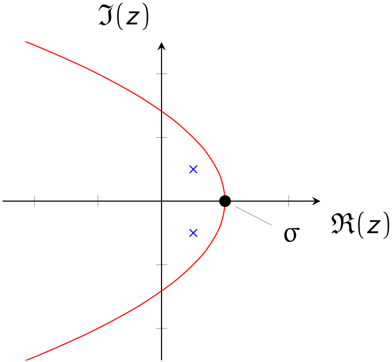
- Parabolic contour:

$$s = \mu(iu+1)^2, \qquad -\infty < u < \infty$$



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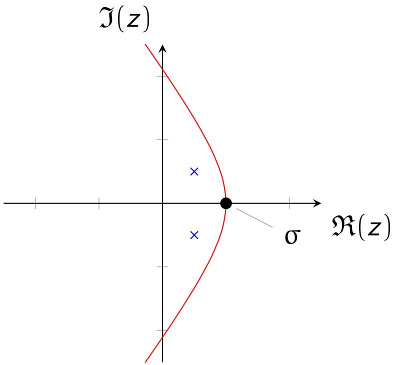

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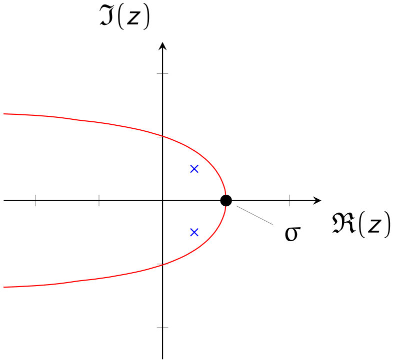

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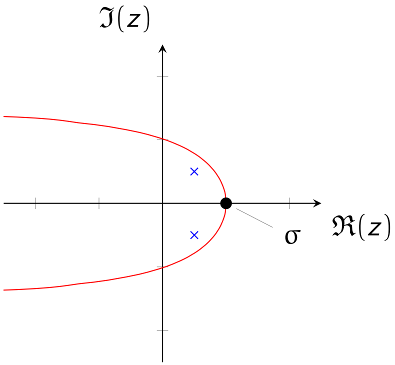

Selecting **contour** and **parameters** depends on the **error analysis**.

Inverting the Laplace Transform

All the contours exploit the fact that e^{st} decays rapidly as $\Re(s) \to -\infty$,

- **Trapezoidal rule** for integral on the real line for which the integrand decay sufficiently rapidly is **exponential**:

Theorem (Trefethen and Weideman 2014, Theorem 5.1)

Suppose that w is analytic in the strip $|\Im(x)| < a$ for some $a > 0$. Suppose further that $w(x) \to 0$ uniformly as $|x| \to +\infty$ in the strip, and that for some M it satisfies

$$\int_{-\infty}^{+\infty} |w(x+ib)| \,\mathrm{d}x \leq M, \quad \forall b \in (-a, a),$$

then for any $h > 0$, the trapezoidal rule $w_{h,N}$ with step-size h exists and satisfies

$$|w_h - \int_{-\infty}^{+\infty} w(x)\,\mathrm{d}x| \leq \frac{2M}{\exp({}^{2\pi a}/_{h}) - 1},$$

and the quantity $2M$ on the numerator is as small as possible.

Inverting the Laplace Transform

- All the contours exploit the fact that e^{st} decays rapidly as $\mathfrak{R}(s) \to -\infty$,
  - **Trapezoidal rule** for integral on the real line for which the integrand decay sufficiently rapidly is **exponential**:

Steepest descent contours

For some functions it is possible to use a technique called "saddle point technique" from complex analysis to estimate the asymptotic of complex integrals. This determines the optimal steepest descent contour.

References for the general problem are:

Talbot: Dingfelder and Weideman 2015; Trefethen, Weideman, and Schmelzer 2006; Weideman 2006,

Parabolic & Hyperbolic: Weideman and Trefethen 2007.

Our case: we've got poles and a branch cut

In our case the function for which we can compute the *Laplace transform* is

$$e_{\alpha,\beta}(t;\lambda) = t^{\beta-1}E_{\alpha,\beta}(t^{\alpha}\lambda), \quad t \in \mathbb{R}_{+}, \qquad \lambda \in \mathcal{C}.$$

That is given by

$$\mathcal{E}_{\alpha,\beta}(t;\lambda) = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \quad \Re(s) > 0, \qquad |\lambda s^{-\alpha}| < 1.$$

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- We have also the **poles** for $\theta = \arg(\lambda)$

$$\bar{s}_j^* = \lambda^{1/\alpha} = |\lambda|^{1/\alpha} e^{i\frac{\theta+2\pi j}{\alpha}}, \quad \left\{ j \in \mathbb{Z} \,\middle|\, -\frac{\alpha}{2} - \frac{\theta}{2\pi} < j \leq \frac{\alpha}{2} - \frac{\theta}{2\pi} \right\},$$

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⚠ There could be lots of poles! Finding suitable contours is difficult.

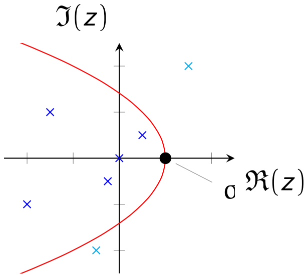
Cauchy's residue theorem to the rescue

We can use Cauchy's residue theorem if we have too many poles

$$e_{\alpha,\beta}(t;\lambda) = \sum_{s^* \in \mathcal{S}^*_{\mathcal{C}}} \mathrm{Res}(e^{st}\mathcal{E}_{\alpha,\beta}(s;\lambda), s^*) + \frac{1}{2\pi i}\int_{\mathcal{C}} e^{st}\mathcal{E}_{\alpha,\beta}(s;\lambda)\,\mathrm{d}s.$$

- $\mathcal{S}^*_{\mathcal{C}}$ is the set of all singularities lying on the rightmost part of the complex plane delimited by $\mathcal{C}$,
- We can compute the residual in close form:

$$\mathrm{Res}(e^{st}\mathcal{E}_{\alpha,\beta}(s;\lambda), s^*) = \frac{1}{\alpha}(s^*)^{1-\beta}e^{s^*t}.$$



The full algorithm (Garrappa 2015)

To build the full algorithm few technical steps are needed:

1. Finding an ordering of the poles,

$$\phi(s) = \frac{\mathfrak{R}(s) + |s|}{2}, \qquad 0 = \phi(s_0^*) < \phi(s_1^*) < \cdots < \phi(s_J^*),$$

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</> it.mathworks.com/matlabcentral/fileexchange/48154-the-mittag-leffler-function

Summary and anticipations

We did

- Uncovered properties of Riemann-Liouville Derivatives,
- Introduced the Caputo Derivative,
- Formulation, existence and uniqueness results for FODEs,
- The Mittag-Leffler function and its computation.

Next up

- Numerical methods for the integration of FODEs.

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