From FEM Discretizations to Saddle-Point Matrices

Iterative Methods for Large-Scale Saddle-Point Problems

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George Pólya (1887–1985)

*"In order **to solve** this differential equation you look at it till a solution occurs to you."*

How to Solve It (Princeton 1945)

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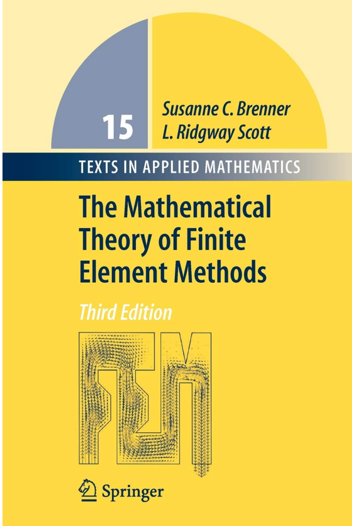
How to Solve It (Princeton 1945)

We are gonna settle for **approximating its solution**.

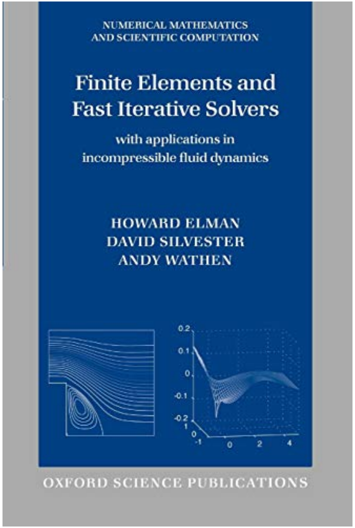
Overview

The main sources



S. C. Brenner and L. R. Scott (2008). *The mathematical theory of finite element methods*. Third. Vol. 15. Texts in Applied Mathematics. Springer, New York, pp. xviii+397. ISBN: 978-0-387-75933-3

H. C. Elman, D. J. Silvester, and A. J. Wathen (2014). *Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics*. Second. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, pp. xiv+479. ISBN: 978-0-19-967880-8



Basic Concepts

Consider the **two-point boundary value problem** (BVP):

$$\begin{cases} -\dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f, & \text{in } (0,1), \\ u(0) = 0, & u'(1) = 0. \end{cases}$$

If u is the solution and $v \in V$ is a sufficiently regular for which $v(0) = 0$, then **integration by parts** yields:

$$\begin{aligned} (f, v) = \int_0^1 f(x)v(x)\,\mathrm{d}x &= -\int_0^1 u''(x)v(x)\,\mathrm{d}x \\ &= \int_0^1 u'(x)v'(x)\,\mathrm{d}x = a(u,v). \end{aligned}$$

Then **the solution** u to our BVP is characterized by

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Sobolev Spaces: multi-index notation

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First some notation

Given a **multi-index** $\boldsymbol{\alpha} \in \mathbb{N}^n$ we denote with

$$|\boldsymbol{\alpha}| = \sum_{i=1}^{n} \alpha_i,$$

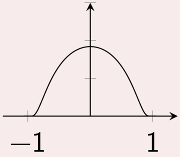
the length of the multi-index. For a function $\varphi \in \mathcal{C}^\infty$, we denote the usual **pointwise partial derivative** by

$$D^{\boldsymbol{\alpha}}\varphi = D^{\boldsymbol{\alpha}}_{\mathbf{x}}\varphi = \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\boldsymbol{\alpha}} \varphi = \varphi^{(\boldsymbol{\alpha})} = \partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\varphi = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \varphi.$$

Sobolev Spaces: building blocks

Definition: compact support functions

Let $\Omega \subseteq \mathbb{R}^n$ a domain. We denote by $\mathcal{D}(\Omega)$ or $\mathcal{C}_0^\infty(\Omega)$ the set of $\mathcal{C}^\infty(\Omega)$ **functions with compact support** in Ω, i.e., the $\mathcal{C}^\infty(\Omega)$ functions for which the closure of the set of the points in which they are not zero is compact in Ω.



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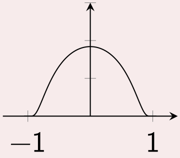

Definition: locally integrable functions

Given a domain Ω we define the set of **locally integrable functions** as

$$\mathbb{L}^1_{\text{loc}}(\Omega) = \{f \,:\, f \in \mathbb{L}^1(K)\, \forall K \subset \mathring{\Omega}\; K \text{ compact}\}.$$

Sobolev Spaces: *weak* derivatives

Definition: weak derivative

We say that a function $f \in \mathbb{L}^1_{\text{loc}}(\Omega)$ has a **weak derivative**, $D^\alpha_w f$ provided that there exists a function $g \in \mathbb{L}^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega g(x)\varphi(x)\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega f(x)\varphi^{(\alpha)}(x)\mathrm{d}x, \qquad \forall\, \varphi \in \mathcal{C}_0^\infty(\Omega).$$

If such g exists then we define $D^\alpha_w f = g$.

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A couple of **examples**:

- $f(x) = 1 - |x|$ admits as first weak derivative $D^1_w f = g = \chi_{x<0} + \chi_{x>0}$,
- If $f \in \mathcal{C}^{|\alpha|}(\Omega)$ for an arbitrary α, then $D^\alpha_w f = D^\alpha f$.

Sobolev space

Definition: Sobolev norms and spaces

Let $k \in \mathbb{N}$, $f \in \mathbb{L}^1_{\text{loc}}(\Omega)$, suppose that the weak derivative $D^{\alpha}_w f$ exists for all $|\alpha| \leq k$. We define the **Sobolev norm**

$$\|f\|_{W^k_p(\Omega)} = \begin{cases} \left(\sum_{\alpha : |\alpha| \leq k} \|D^{\alpha}_w f\|^p_{\mathbb{L}^p(\Omega)} \right)^{\frac{1}{p}}, & 1 \leq p < +\infty \\ \max_{\alpha : |\alpha| \leq k} \|D^{\alpha}_w f\|_{\mathbb{L}^{\infty}(\Omega)}, & p = \infty. \end{cases}$$

We define the **Sobolev space** $W^k_p(\Omega)$ as

$$W^k_p(\Omega) = \left\{ f \in \mathbb{L}^1_{\text{loc}}(\Omega) \, : \, \|f\|_{W^k_p(\Omega)} < \infty \right\}.$$

Sobolev space: a collection of results

Theorem(s)

(i) The Sobolev space $W_p^k(\Omega)$ is a Banach space,

(ii) Let Ω be any open set, then $\mathcal{C}^\infty(\Omega) \cap W_p^k(\Omega)$ is dense in $W_p^k(\Omega)$ for $p < \infty$,

(iii) $k, m \in \mathbb{N}$, $k \leq m$, $1 \leq p \leq \infty \Rightarrow W_p^m(\Omega) \subset W_p^k(\Omega)$,

(iv) Ω bounded, $k \in \mathbb{N}$, $1 \leq p \leq q \leq \infty \Rightarrow W_q^k(\Omega) \subset W_p^k(\Omega)$,

Sobolev space: a collection of results

Theorem(s)

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Definition: Lipschitz boundary

Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^\varkappa$. Ω is a **Lipschitz domain** if $\forall\, p \in \partial\Omega$ exists a hyperplane H of dimension $n-1$ through p, a Lipschitz-continuous function $g : H \to \mathbb{R}$ over that hyperplane, and reals $r > 0$ and $h > 0$ such that

- $\Omega \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H,\ -h < y < g(x)\}$,
- $(\partial\Omega) \cap C = \{x + y\vec{n} \mid x \in B_r(p) \cap H,\ g(x) = y\}$,

where $\vec{n}$ is a unit vector that is normal to H, $B_r(p) := \{x \in \mathbb{R}^n \mid \|x - p\| < r\}$ is the open ball of radius r, $C := \{x + y\vec{n} \mid x \in B_r(p) \cap H,\ -h < y < h\}$.

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(iv) Ω bounded, $k \in \mathbb{N}$, $1 \leq p \leq q \leq \infty \Rightarrow W_q^k(\Omega) \subset W_p^k(\Omega)$,

(v) If $\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary, $\forall\, k \in \mathbb{N}$, $1 \leq p \leq \infty$, there exist $E : W_p^k(\Omega) \to W_p^k(\mathbb{R}^n)$ satisfying $Ev|_\Omega = v \;\forall, v \in W_p^k(\Omega)$, and $\|Ev\|_{W_p^k(\mathbb{R}^n)} \leq C\|v\|_{W_p^k(\Omega)}$ with C independent of v,

(vi) If $\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary, $\forall\, k \in \mathbb{N}$, $1 \leq p < \infty$, and $m < k$, then

$$\exists\, C > 0 \,:\, \forall u \in W_p^k(\Omega)\; \|u\|_{W_\infty^m(\Omega)} \leq C\|u\|_{W_p^k(\Omega)} \begin{cases} k - m \geq n, & p = 1, \\ k - m > \frac{n}{p}, & p > 1. \end{cases}$$

And there exist a function in $\mathcal{C}^m$ in the $\mathbb{L}^p$ equivalence class of u.

Sobolev space: finally we have got an answer!

If you have forgotten the question, we were trying to understand for what V **the solution** u characterized by

$$\text{find } u \in V \text{ such that } a(u, v) = (f, v) \qquad \forall\, v \in V.$$

was a meaningful solution to our initial BVP.

The space

$$V = \{v \in W_2^1(\Omega) \ : \ v(0) = 0\}.$$

By the **extension property** and the **Sobolev inequality** we now know that pointwise values are well defined for functions $W_2^1(\Omega)$.

But all this machinery was needed just to **validate the formulation**, how do we go to a *discrete solution*?

Building a discrete space

To move to a discrete setting, we need to select a **finite subspace** $S \subset V$. With this, we can impose the Ritz-Galerkin conditions:

$$\text{find } u_S \in S \text{ such that } a(u_S, v) = (f, v) \qquad \forall\, v \in S.$$

- Since S is finite-dimensional, there exists a basis $\phi_1, \ldots, \phi_n$ of S,
- Thus, $u_S = \sum_{i=1}^{n} U_i \phi_i \in S$, $U_i \in \mathbb{R}$ for $i = 1, \ldots, n$,
- Ritz-Galerkin conditions are now a **system of linear equations** for the unknown coefficients U_i:

$$K\mathbf{U} = \mathbf{F},$$

  with

  - $\mathbf{U} = (U_1, \ldots, U_n)^T \in \mathbb{R}^n$,
  - $\mathbf{F} = (F_1, \ldots, F_n)^T \in \mathbb{R}^n$, for $F_i = (f, \phi_i)$,
  - $\mathbf{K} = (K_{ij}) \in \mathbb{R}^{n \times n}$, for $K_{ij} = a(\phi_i, \phi_j)$.

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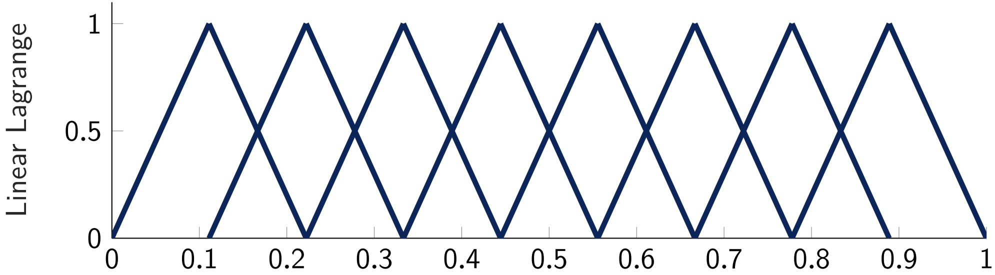
What are examples of such S?

Lagrange basis



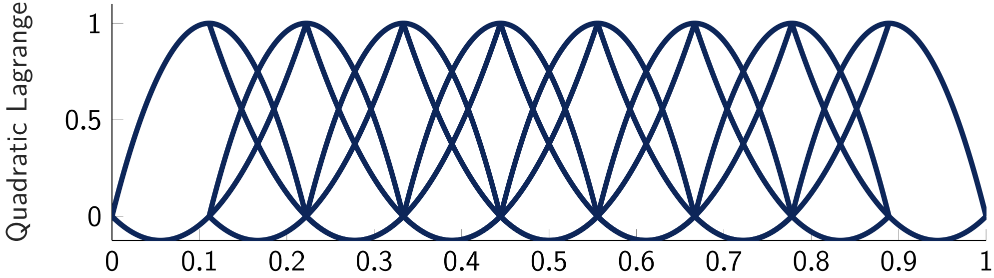
Let $0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$, we consider the linear space of functions $v \in S$ s.t.

(i) $v \in \mathcal{C}^0([0,1])$,

(ii) $v|_{[x_{i-1},x_i]}$ is a linear polynomial, $i = 1, \ldots, n$, and

(iii) $v(0) = 0$.

Lagrange basis



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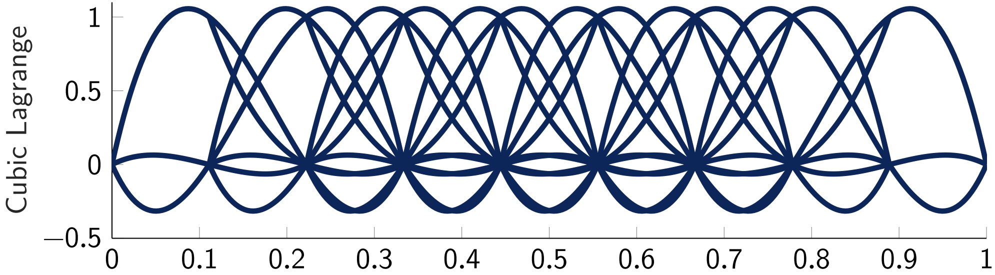

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(i) $v \in \mathcal{C}^0([0,1])$,

(ii) $v|_{[x_{i-3},x_i]}$ is a cubic polynomial, $i = 3, \ldots, n$, and

(iii) $v(0) = 0$.

Convergence and approximation properties

We have a **theoretical framework for solutions**, examples of **discrete spaces**, but what about convergence?

Sobolev meets Hilbert

W_p^k is a Hilbert space for $p = 2$, with inner product

$$\langle f, g \rangle_{W_2^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g).$$

We write: $H^k(\Omega) \equiv W_2^k(\Omega)$, and $H_0^k(\Omega) = \{v \in W_2^k(\Omega) \ : \ v \equiv 0 \text{ on } \partial\Omega\}$.

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Our V space

$$V = H_0^1([0,1]).$$

Convergence and approximation properties

Variational problem

For a given Hilbert space V, a bilinear form $a : V \times V \to \mathbb{R}$ and a linear functional $F : V \to \mathbb{R}$, find $u \in V$ such that:

$$a(u, v) = F(v), \qquad \text{for all } v \in V. \tag{VP}$$

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Theorem (Lax-Milgram).

Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ a bilinear form, and $F : V \to \mathbb{R}$ a linear functional s.t.

- **Coercivity** $\exists\, c_1 > 0$ s.t. $a(v, v) \geq c_1 \|v\|_V^2$ for all $v \in V$.
- **Continuity** $\exists\, c_2, c_3 > 0$ s.t. $a(v, w) \leq c_2 \|v\|_V \|w\|_V$, and $F(v) \leq c_3 \|v\|_V$ for all $v, w \in V$.

$\Rightarrow \exists!\, u \in V$ satisfying (VP), and $\|u\|_V \leq \dfrac{1}{c_1} \|F\|_{V^*}$.

Convergence and approximation properties

In the **conforming Galerkin approach** we chose a (finite-dimensional) closed subspace $V_h \subset V$ and look for $u_h \in V_h$ satisfying

$$a(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h. \tag{VP$_h$}$$

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Theorem

Under the assumptions of the Lax-Milgram, for any closed subspace $V_h \subset V$, there exists a unique solution $u_h \in V_h$ of (VP$_h$) satisfying

$$\|u_h\|_V \leq \frac{1}{c_1} \|F\|_{V^*}.$$

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Céa's lemma

Let u_h be the solution of (VP$_h$) for given $V_h \subset V$ and u be the solution of variational problem (VP). Then,

$$\|u - u_h\|_V \leq \frac{c_2}{c_1} \inf_{v_h \in V_h} \|u - v_h\|_V \,,$$

where c_1 and c_2 are the constants from the **coercivity** and **continuity** assumptions.

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The conforming idea

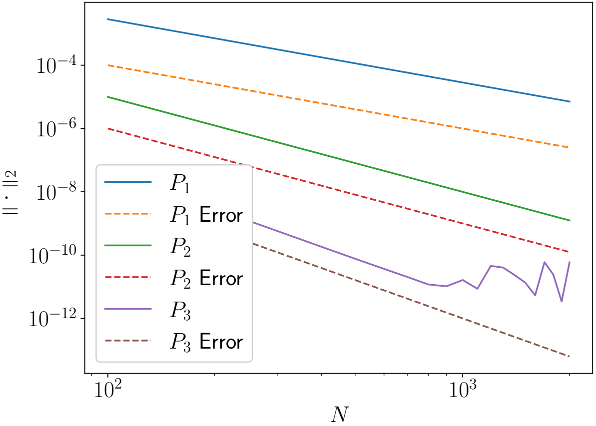
The **error** of the **conforming Galerkin** approach is **determined** by the **approximation error** of the exact solution **in** V_h.

Error estimate on 1D problem

The **test problem** we consider is

$$\begin{cases} -u_{xx} = f(x), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega \end{cases}$$

for $f(x) = 2\cos(x)/e^x$ and $g(x) = \sin(x)/e^x$ on $\Omega = (0, 10)$. We discretize it with Lagrangian 1, 2 and 3 elements and report the computed error: $\|u - u_{\text{ex}}\|_{\mathbb{L}^2(\Omega_h)}$ on the uniform grid with N points.



FEniCSx Code Example

We can implement this simple case in the FEniCSx Library in few lines of code

1. First we need to load some packages

```python
from mpi4py import MPI # Needed for the MPI environment
import numpy as np # The numpy package support
from dolfinx import mesh # Handler for the meshes
from dolfinx import fem # FEM building blocks
from dolfinx.fem import FunctionSpace # FEM Function Spaces
import ufl # Language for building up variational formulations
```

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```
nx = 500
Omegah = mesh.create_interval(comm=MPI.COMM_WORLD, nx=nx,
↪ points=(0,10))
V = FunctionSpace(Omegah, ("CG", 1))
```

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3. Then we need a bit of work to impose **essential boundary conditions**

```
g = fem.Function(V)
g.interpolate(lambda x: np.sin(x[0])/np.exp(x[0]))
tdim = Omegah.topology.dim
fdim = tdim - 1
Omegah.topology.create_connectivity(fdim, tdim)
boundary_facets =
↪  np.flatnonzero(mesh.compute_boundary_facets(Omegah.topology))
boundary_dofs = fem.locate_dofs_topological(V, fdim,
↪  boundary_facets)
bc = fem.dirichletbc(g, boundary_dofs)
```

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4. We create **test** and **trial functions**

```
u = ufl.TrialFunction(V)
v = ufl.TestFunction(V)
```

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5. We build the source and the **variational formulation**

```
f = fem.Function(V)
f.interpolate(lambda x: 2.0*np.cos(x[0])/np.exp(x[0]))
a = ufl.dot(ufl.grad(u), ufl.grad(v)) * ufl.dx
F = f * v * ufl.dx
```

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6. Finally we **solve** the **linear system** (directly...it's 1D!)

```
problem = fem.petsc.LinearProblem(a, F, bcs=[bc],
↪  petsc_options={"ksp_type": "preonly", "pc_type": "lu"})
uh = problem.solve()
```

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7. and **compute the error**: `Error_L2 : 1.14e-04`

```
V2 = fem.FunctionSpace(Omegah, ("CG", 2))
uex = fem.Function(V2)
uex.interpolate(lambda x: np.sin(x[0])/np.exp(x[0]))
L2_error = fem.form(ufl.inner(uh - uex, uh - uex) * ufl.dx)
error_local = fem.assemble_scalar(L2_error)
error_L2 = np.sqrt(Omegah.comm.allreduce(error_local, op=MPI.SUM))
```

FEniCSx Code Example

We can implement this simple case in the FEniCSx Library in few lines of code

1. First we need to load some packages
2. Then we **build** the **mesh** and the **function space**
3. Then we need a bit of work to impose **essential boundary conditions**
4. We create **test** and **trial functions**
5. We build the source and the **variational formulation**
6. Finally we **solve** the **linear system** (directly...it's 1D!)
7. and **compute the error**: `Error_L2 : 1.14e-04`

To run the example there is a Python notebook using FEniCSx shared through bit.ly/3tTEBfl (and executed on Google Colab).

FEM Spaces

We can build many different types of **F**inite **E**lements.

FE Definition (Ciarlet, 1978)

A *finite element* is a triple $(K, \mathcal{P}, \mathcal{N})$ where

(i) $K \subset \mathbb{R}^n$ is a simply connected bounded open set with piecewise smooth boundary (*element domain*);

(ii) $\mathcal{P}$ is a finite-dimensional space of functions defined on K (*space of shape functions*);

(iii) $\mathcal{N} = \{N_1, \dots, N_d\}$ is a basis of $\mathcal{P}^*$ (*degrees of freedom*).

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(iii) $\mathcal{N} = \{N_1, \ldots, N_d\}$ is a basis of $\mathcal{P}^*$ (*degrees of freedom*).

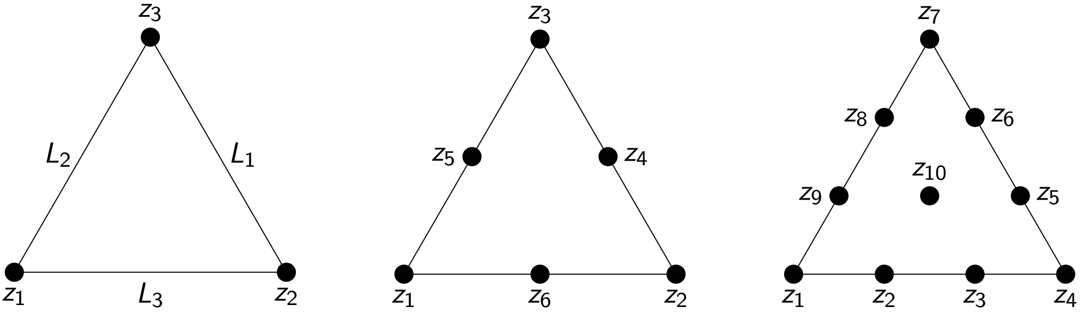
Dual basis definition

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. A basis $\{\psi_1, \ldots, \psi_d\}$ of $\mathcal{P}$ is called *dual basis* or *nodal basis* to $\mathcal{N}$ if $N_i(\psi_j) = \delta_{ij}$.

A lineup of some usual (and unusual) suspects

FEM Spaces: triangular finite elements

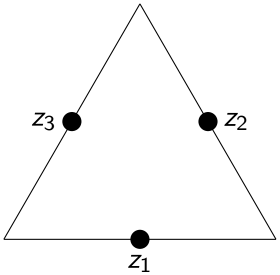
K any triangle, space $\mathcal{P}_k$ of bivariate polynomials of degree $\leq k$,



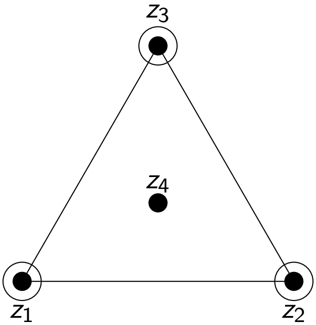
Linear Lagrange element — Quadratic Lagrange element — Cubic Lagrange element

"●" Point evaluations determining the $\mathcal{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_{\frac{1}{2}(k+1)(k+2)}\}$.

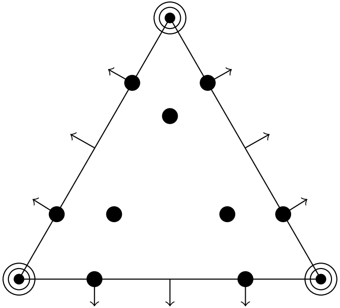
FEM Spaces: triangular finite elements



Linear nonconforming
Crouzeix-Raviart element



Cubic Hermite element

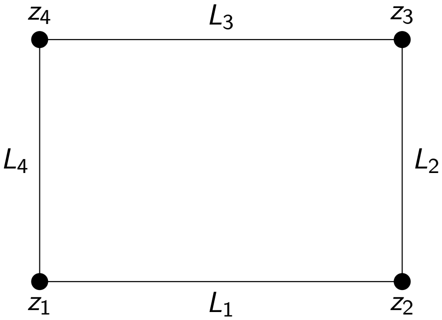


Quintic Argyris element

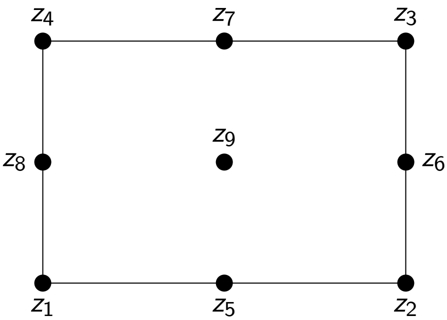
● Point evaluations, ⊙ Gradient evaluations, ◎ Three second derivative, $\uparrow$ Normal derivative.

FEM Spaces: rectangular finite elements

K any rectangle, space $\mathcal{Q}_k = \left\{ \sum_j c_j p_j(x) q_j(x),\ p_j, q_j \in \mathbb{P}_{\leq k}[x] \right\}$,



Bilinear Lagrange element



Biquadratic Lagrange element

- Point evaluations for $\mathcal{N} = \{N_1, \ldots, N_d\}$, $d = \dim \mathcal{Q}_k = (\dim \mathbb{P}_{\leq k}[x])^2$.

Periodic Table of the Finite Elements

$\mathcal{P}_r^-\Lambda^k$

The shape function space for $\mathcal{P}_r^-\Lambda^k$ is
$$\mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{P}_{r-1}\Lambda^{k+1},$$
where κ is the Koszul differential.[7] It includes the full polynomial space $\mathcal{P}_{r-1}\Lambda^k$, is included in $\mathcal{P}_r\Lambda^k$, and has dimension
$$\dim \mathcal{P}_r^-\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}.$$
The degrees of freedom are given on faces f of dimension $d \geq k$ by moments of the trace weighted by a full polynomial space:
$$u \mapsto \int_f (\mathrm{tr}_f\, u)\wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f).$$
The spaces with constant degree r form a complex:
$$\mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n.$$

$\mathcal{P}_r\Lambda^k$

The shape function space for $\mathcal{P}_r\Lambda^k$ consists of all differential k-forms with polynomial coefficients of degree at most r, and has dimension
$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}.$$
The degrees of freedom are given on faces f of dimension $d \geq k$ by moments of the trace weighted by a $\mathcal{P}^-$ space:
$$u \mapsto \int_f (\mathrm{tr}_f\, u)\wedge q, \quad q \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f).$$
The spaces with decreasing degree r form a complex:
$$\mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n.$$

$\mathcal{Q}_r^-\Lambda^k$

This family is constructed from the complex of 1-dimensional finite elements using a tensor product construction.[10] The shape function space on the unit cube $\square_n = I^n$ is given by
$$\bigoplus_{\sigma\in\Sigma(k,n)}\left[\bigotimes_{i=1}^{n}\mathcal{P}_{r-\lambda_{i,\sigma}}(I)\right] dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k},$$
where $\Sigma(k,n)$ denotes the increasing maps $\{1,\ldots,k\} \to \{1,\ldots,n\}$. Its dimension is $\dim \mathcal{Q}_r^-\Lambda^k(\square_n) = \binom{n}{k}r^k(r+1)^{n-k}$. The degrees of freedom are
$$u \mapsto \int_f (\mathrm{tr}_f\, u)\wedge q, \quad q \in \mathcal{Q}^-_{r-1}\Lambda^{d-k}(f).$$
The spaces with constant degree r form a complex:
$$\mathcal{Q}_r^-\Lambda^0 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{Q}_r^-\Lambda^n.$$

$\mathcal{S}_r\Lambda^k$

The shape function space for $\mathcal{S}_r\Lambda^k$ is given by
$$\mathcal{P}_r\Lambda^k \oplus \bigoplus_{\ell\geq 1}[\kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1} \oplus d\kappa\mathcal{H}_{r+\ell,\ell}\Lambda^k],$$
where $\mathcal{H}_{r,\ell}\Lambda^k$ consists of homogeneous polynomial k-forms of degree r which are linear and undifferentiated in at least ℓ variables.[11] Its dimension is $\dim \mathcal{S}_r\Lambda^k(\square_n) = \sum_{d\geq k} 2^{n-d}\binom{n}{d}\binom{r-d+2k}{d}\binom{d}{k}$. The degrees of freedom are
$$u \mapsto \int_f (\mathrm{tr}_f\, u)\wedge q, \quad q \in \mathcal{P}_{r-2(d-k)}\Lambda^{d-k}(f).$$
The spaces with decreasing degree r form a complex:
$$\mathcal{S}_r\Lambda^0 \xrightarrow{d} \mathcal{S}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}_{r-n}\Lambda^n.$$

Legend

Finite elements

The table presents the primary spaces of finite elements for the discretization of the fundamental operators of vector calculus: the gradient, curl, and divergence. A finite element space is a space of piecewise polynomial functions on a domain determined by: (1) a mesh of the domain into polyhedral cells called *elements*, (2) a finite dimensional space of polynomial functions on each element called the *shape functions*, and (3) a unisolvent set of functionals on the shape functions of each element called *degrees of freedom* (DOFs), each DOF being associated to a (generalized) face of the element, and specifying a quantity which takes a single value for all elements sharing the face. The element diagrams depict the DOFs and their association to faces.

The spaces $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ depicted on the left half of the table are the two primary families of finite element spaces for meshes of simplices, and the spaces $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$ on the right side are for meshes of cubes or boxes. Each is defined in any dimension $n \geq 1$ for each value of the polynomial degree $r \geq 1$, and each value of $0 \leq k \leq n$. The parameter k refers to the operator: the spaces consist of differential k-forms which belong to the domain of the kth exterior

References

FEM Spaces: it's a vast world

Much of what we discussed and of what we are going to discuss in the next slides can be applied to *FEM-adjacent* methods, a (obviously not exhaustive) list of ideas:

- DG: Discontinuous Galerkin, (Cockburn, Karniadakis, and Shu 2000) for a general overview, linear solvers (Ayuso de Dios et al. 2014; Dobrev et al. 2006)...
- IgA: Isogeometric Analysis, (Cottrell, Hughes, and Bazilevs 2009) for a general overview, adaptive meshes (Giannelli, Jüttler, and Speleers 2012; Patrizi and Dokken 2020), linear solvers (Donatelli et al. 2015; Horníková, Vuik, and Egermaier 2021; Sangalli and Tani 2016)...
- VEM: Virtual Elements, (Beirão da Veiga et al. 2014, 2016) for a general overview, linear solvers (Antonietti, Mascotto, and Verani 2018; Dassi and Scacchi 2020)...

Another nice source of information is: defelement.com.

Variational crimes

“The crime is now logical and reasonable.”

Murder for Christmas, A. Christie

The Penal Code

- *Petrov–Galerkin* approaches, where the function u satisfying $a(u,v)$ for all $v \in V$ is an element of $U \neq V$;
- *non-conforming* approaches, where the discrete spaces U_h and V_h are not subspaces of U and V, respectively; and
- *non-consistent* approaches, where the discrete problem involves a bilinear form $a_h \neq a$ (and a_h might not be well-defined for all $u \in U$).

We thus need a **more general framework** that covers these cases as well.

- U, V be Banach spaces, with V reflexive, U^*, V^* denote their topological duals
- Given $a : U \times V \to \mathbb{R}$ bilinear, $F \in V^*$ continuous we look for $u \in U$ satisfying

$$a(u,v) = F(v) \quad \text{for all } v \in V. \tag{$\mathcal{W}$}$$

Existence and uniqueness in a world full of crimes

Theorem Banach–Nečas–Babuška

Let U and V be Banach spaces and V be reflexive. If $a : U \times V \to \mathbb{R}$ and $F : V \to \mathbb{R}$ satisfy:

(i) *Inf–sup condition*: there exists a $c_1 > 0$ such that

$$\inf_{u \in U} \sup_{v \in V} \frac{a(u,v)}{\|u\|_U \|v\|_V} \geq c_1.$$

(ii) *Continuity*: there exist c_2, c_3 such that

$$|a(u,v)| \leq c_2 \|u\|_U \|v\|_V , \quad |F(v)| \leq c_3 \|v\|_V , \ \forall\, u \in U, \ \forall\, v \in V$$

(iii) *Injectivity*: for any $v \in V$ $a(u,v) = 0$ for all $u \in U$ $\quad$ implies $\quad v = 0$.

Then there exists a unique solution $u \in U$ to $(\mathcal{W})$, which satisfies

$$\|u\|_U \leq \frac{1}{c_1} \|F\|_{V^*} .$$

Mixed Methods - The Poisson equation

Let us start again from the Poisson equation with homogeneous Dirichlet conditions

$$\begin{cases} -\Delta u = -\nabla \cdot \nabla u = -\nabla^2 u = -\operatorname{div}\operatorname{grad} u = f, & \mathbf{x} \in \Omega \subset \mathbb{R}^n \\ u = 0, & \mathbf{x} \in \partial\Omega. \end{cases}$$

We introduce an **auxiliary variable** $\sigma = \nabla u \in \mathbb{L}^2(\Omega)^n$ and rewrite it as

$$\begin{cases} \nabla u - \sigma & = 0, \\ -\nabla \cdot \sigma & = f. \end{cases}$$

This system can be formulated in variational form in two different ways:

1. we *formally* integrate by parts in the second equation $\Rightarrow$ *primal* approach,
2. we *formally* integrate by parts in the first equation $\Rightarrow$ *dual* approach.

Mixed Methods - The Poisson equation - Primal

We look for $(\sigma, u) \in \mathbb{L}^2(\Omega)^n \times \mathbb{H}_0^1(\Omega)$ satisfying

$$\begin{cases} (\sigma,\tau) - (\tau, \nabla u) = 0 & \text{for all } \tau \in \mathbb{L}^2(\Omega)^n, \\ -(\sigma, \nabla v) = -(f, v) & \text{for all } v \in \mathbb{H}_0^1(\Omega). \end{cases}$$

that we can restate in **abstract form** as

$$a(\sigma,\tau) = (\sigma,\tau) : V \times V \to \mathbb{R}, \qquad b(v,\mu) = -(v, \nabla\mu) : V \times M \to \mathbb{R},$$

on the two (reflexive) Banach spaces $V = \mathbb{L}^2(\Omega)^n$ and $M = \mathbb{H}_0^1(\Omega)$ for the problem

$$\text{Find } u, \lambda \text{ s.t. } \begin{cases} a(u, v) + b(v, \lambda) = \langle f, v\rangle_{V^*,V} & \text{for all } v \in V, \\ b(u, \mu) = \langle g, \mu\rangle_{M^*,M} & \text{for all } \mu \in M. \end{cases}$$

Mixed Methods - Abstract Saddle-Point formulation

To uncover the connection with the discrete case we are aiming at, let us reformulate the previous in operator form by introducing

$$\begin{aligned}
&A : V \to V^*, && \langle Au, v\rangle_{V^*,V} = a(u,v) && \text{for all } v \in V,\\
&B : V \to M^*, && \langle Bu, \mu\rangle_{M^*,M} = b(u,\mu) && \text{for all } \mu \in M,\\
&B^* : M \to V^*, && \langle B^*\lambda, v\rangle_{V^*,V} = b(v,\lambda) && \text{for all } v \in V.
\end{aligned}$$

From which we rewrite our problem as

$$\text{Find } u, \lambda \text{ s.t. } \begin{cases} Au + B^*\lambda = f & \text{in } V^*,\\ Bu = g & \text{in } M^*. \end{cases}$$

At this stage, this should be very familiar!

Mixed Methods - Abstract Saddle-Point formulation

Abstract Saddle-Point

$$\text{Find } u, \lambda \text{ s.t. } \begin{cases} Au + B^*\lambda = f & \text{in } V^*, \\ Bu = g & \text{in } M^*. \end{cases}$$

- If B is *invertible* $\Rightarrow$ **existence** and **uniqueness** first of u and then of λ follow immediately,

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- Usually, we are not this lucky, remember our **starting example**:

$$< B\sigma, \mu >_{(\mathbb{H}^1_0)^*, \mathbb{H}^1_0} = b(\sigma, \mu) = -(\sigma, \nabla\mu),$$

that is we have to **require** that A is **injective** and **coercive** on $\ker B$ to obtain a **unique** u...

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  that is we have to **require** that A is **injective** and **coercive** on $\ker B$ to obtain a **unique** u...
- ...for the **existence** of λ we then need B^* to be **surjective**.

Mixed Methods - Banach–Nečas–Babuška

Theorem (Continuous Brezzi)

We assume that

(i) $a : V \times V \to \mathbb{R}$ satisfies the conditions of the Banach–Nečas–Babuška Theorem for $U = V = \ker B$

(ii) $b : V \times M \to \mathbb{R}$ is such that the **L**adyzhenskaya–**B**abuška–**B**rezzi condition holds

$$\exists\, \beta > 0 \ : \ \inf_{\mu \in M} \sup_{v \in V} \frac{b(v,\mu)}{\|v\|_V \|\mu\|_M} \geq \beta.$$

$\Rightarrow \exists!\ (u, \lambda) \in V \times M$ solving the mixed saddle-point system and satisfying

$$\|u\|_V + \|\lambda\|_M \leq C(\|f\|_{V^*} + \|g\|_{M^*}).$$

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$a(u, v)$ has to satisfy the BNB condition only on $\ker B$, not on all of V!

LBB condition couples V and M spaces, this is going to have repercussions in a moment!

Mixed Methods - Back to Poisson

Mixed Continuous Primal Poisson Problem

Find $(\sigma, u) \in \mathbb{L}^2(\Omega)^n \times \mathbb{H}^1_0(\Omega)$ s.t.

$$\begin{cases} (\sigma, \tau) - (\tau, \nabla u) = 0 & \forall\, \tau \in \mathbb{L}^2(\Omega)^n, \\ \qquad -(\sigma, \nabla v) = -(f, v) & \forall\, v \in \mathbb{H}^1_0(\Omega). \end{cases}$$

Coercivity: a is coercive on V with constant $\alpha = 1$,

LBB: chose $v \in \mathbb{H}^1_0(\Omega) = M$ and take $\tau = -\nabla v \in \mathbb{L}^2(\Omega)^n = V$, then

$$\sup_{\tau \in V} \frac{b(\tau, v)}{\|\tau\|_V} = \sup_{\tau \in V} \frac{-(\tau, \nabla v)}{\|\tau\|_{\mathbb{L}^2(\Omega)^n}} \geq \frac{(\nabla v, \nabla v)}{\|\nabla v\|_{\mathbb{L}^2(\Omega)^n}} = |v|_{\mathbb{H}^1} \geq c_\Omega^{-1} \|v\|_M$$

to get C_Ω^{-1} we use Poincaré inequality: for $1 \leq p < \infty$, Ω an open bounded set $\Rightarrow$

$$\exists\, c_\Omega : \|f\|_{W^1_p(\Omega)} \leq c_\Omega |f|_{W^1_p(\Omega)} \text{ depending only on } p \text{ and } \Omega.$$

Mixed Methods - Back to Poisson

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Mixed Methods - Galerkin Approach

Abstract Saddle-Point

$$\text{Find } u, \lambda \text{ s.t. } \begin{cases} Au + B^*\lambda = f & \text{in } V^*, \\ Bu = g & \text{in } M^*. \end{cases}$$

- $V_h \subset V$, $M_h \subset M$,
- V_h and M_h **cannot be** selected independently!

Mixed Methods - Galerkin Approach

~~Abstract~~ Discrete Saddle-Point

Find u_h, λ_h such that

$$\begin{cases} a(u_h, v_h) + b(v_h, \lambda_h) = \langle f, v_h \rangle_{V^*,V} \ \forall\, v_h \in V_h, \\ b(u_h, \mu_h) = \langle g, \mu_h \rangle_{M^*,M} \ \forall\, \mu_h \in M_h. \end{cases}$$

- $V_h \subset V$, $M_h \subset M$,
- V_h and M_h **cannot be** selected independently!

Theorem (Discrete Brezzi)

$$\text{If } \exists \alpha_h > 0 : \inf_{u_h \in \ker B_h} \sup_{v_h \in \ker B_h} \frac{a(u_h, v_h)}{\|u_h\|_V \|v_h\|_V} \geq \alpha_h,$$

$$\text{If } \exists \beta_h > 0 : \inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, \mu_h)}{\|v_h\|_V \|\mu_h\|_M} \geq \beta_h.$$

$\Rightarrow \exists!\ (u_h, \lambda_h) \in V_h \times M_h$ solving the discrete saddle-point and satisfying

$$\|u_h\|_V + \|\lambda_h\|_M \leq C_h(\|f\|_{V^*} + \|g\|_{M^*}).$$

Mixed Methods - Dual approach

We integrate by parts the **first equation**:

$$\int_{\Omega} (\mathrm{div}\, \tau) w \, \mathrm{d}x + \int_{\Omega} \tau \cdot \nabla w \, \mathrm{d}x = \int_{\partial\Omega} (\tau \cdot \nu) w \, \mathrm{d}x$$

We need to define the proper Sobolev space.

Mixed Methods - Dual approach

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$$\int_\Omega (\operatorname{div}\tau) w \,\mathrm{d}x + \int_\Omega \tau \cdot \nabla w \,\mathrm{d}x = \int_{\partial\Omega} (\tau \cdot \nu) w \,\mathrm{d}x$$

We need to define the proper Sobolev space.

$\mathbb{H}(\operatorname{div})$

We define the space

$$\mathbb{H}(\operatorname{div}) = \{\tau \in \mathbb{L}^2(\Omega)^n \,:\, \operatorname{div}\tau \in \mathbb{L}^2(\Omega)\},$$

with the norm

$$\|\tau\|^2_{\mathbb{H}(\operatorname{div})} := \|\tau\|^2_{\mathbb{L}^2(\Omega)^n} + \|\operatorname{div}\tau\|^2.$$

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$$\|\tau\|^2_{\mathbb{H}(\operatorname{div})} := \|\tau\|^2_{\mathbb{L}^2(\Omega)^n} + \|\operatorname{div}\tau\|^2.$$

Well posedness of the *normal trace*.

$\mathcal{C}^\infty(\overline{\Omega})^n$ is dense in $\mathbb{L}^2(\Omega)^n \supset \mathbb{H}(\operatorname{div}) \Rightarrow \tau \in \mathbb{H}(\operatorname{div})$ has $(\tau|_{\partial\Omega}\cdot\nu) \in \mathbb{H}^{-1/2}(\partial\Omega)$.

Mixed Methods - Dual approach

The **dual problem** is then

$$\text{find } (\sigma, u) \in \mathbb{H}(\mathrm{div}) \times \mathbb{L}^2 \text{ s.t. } \begin{cases} (\sigma, \tau) + (\mathrm{div}\, \tau, u) = 0 & \forall\, \tau \in \mathbb{H}(\mathrm{div}), \\ (\mathrm{div}\, \sigma, v) = -(f, v) & \forall v \in \mathbb{L}^2. \end{cases}$$

- we have used that $u|_{\partial\Omega} = 0$,

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- we have used that $u|_{\partial\Omega} = 0$,
- This is a **general saddle problem** with $V = \mathbb{H}(\mathrm{div})$, and $M = \mathbb{L}^2$:

$$a(\sigma, \tau) = (\sigma, \tau), \qquad b(\sigma, v) = (\mathrm{div}\, \sigma, v).$$

and a and b bounded by Cauchy–Schwarz inequality.

Mixed Methods - Dual approach

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$$a(\sigma, \tau) = (\sigma, \tau), \qquad b(\sigma, v) = (\mathrm{div}\,\sigma, v).$$

  and a and b bounded by Cauchy–Schwarz inequality.
- For the existence of the solution we need to prove **coercivity** for a.

$$\ker B = \{\tau \in \mathbb{H}(\mathrm{div}) \;:\; (\mathrm{div}\,\tau, v) = 0\;\forall\, v \in \mathbb{L}^2\}$$

  Since $\mathrm{div}\,\tau \in \mathbb{L}^2$ we have $\|\mathrm{div}\,\tau\|_{\mathbb{L}^2} = 0$ whenever $\tau \in \ker B \subset \mathbb{H}(\mathrm{div})$, and therefore

$$a(\tau, \tau) = \|\tau\|^2_{\mathbb{L}^2(\Omega)^n} = \|\tau\|^2_{\mathbb{H}(\mathrm{div})} \quad \forall\, v \in \ker B,$$

  indeed we have just proved **coercivity with** $\alpha = 1$.

Mixed Methods - Dual approach

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$$a(\sigma,\tau) = (\sigma,\tau), \qquad b(\sigma, v) = (\mathrm{div}\,\sigma, v).$$

  and a and b bounded by Cauchy–Schwarz inequality.
- The form a is coercive on $\ker B$ with $\alpha = 1$,
- We now need to verify the LBB condition. This requires some work.

Mixed Methods - Dual approach - LBB

Assumption:

We make the simplifying assumption of having $\partial\Omega$ represented by a $\mathcal{C}^1$ function or, analogously, having Ω convex.

Lemma (Surjectivity)

For any $f \in \mathbb{L}^2$, there exists a function $\tau \in \mathbb{H}(\mathrm{div})$ with $\operatorname{div}\tau = f$ and $\|\tau\|_{\mathbb{H}(\mathrm{div})} \leq C\|f\|_{\mathbb{L}^2}$.

- The space $\mathbb{H}^1(\Omega)^n \subset \mathbb{H}(\mathrm{div})$, thus if we take a $v \in M = \mathbb{L}^2$ and the corresponding $\tau_v \in \mathbb{H}(\mathrm{div})$ given by the surjectivity lemma (i.e., $\operatorname{div}\tau_v = v$) we find

$$\sup_{\tau\in V} \frac{b(\tau, v)}{\|\tau\|_V} = \sup_{\tau\in V} \frac{(\operatorname{div}\tau, v)}{\|\tau\|_{\mathbb{H}(\mathrm{div})}} \geq \frac{(\tau_v, v)}{\|\tau_v\|_{\mathbb{H}(\mathrm{div})}} \geq \frac{(v, v)}{C\|v\|_{\mathbb{L}^2(\Omega)}} = \frac{1}{c}\|v\|_{\mathbb{L}^2(\Omega)}.$$

We have then proved the LBB condition for $\beta = \frac{1}{C}$.

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- We have then proved the LBB condition for $\beta = \frac{1}{C}$.
- By Continuous Brezzi we have $\exists!\,(\sigma, u) \in V \times M$ solving the **saddle problem** and such that

$$\|\sigma\|_{\mathbb{H}(\mathrm{div})} + \|u\|_{\mathbb{L}^2(\Omega)} \leq C\|f\|_{\mathbb{L}^2(\Omega)}.$$

Mixed Methods - Dual approach - LBB

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- The solution u we have obtained seem to be only in $\mathbb{L}^2$...but u satisfies

$$(\sigma, \tau) + (\mathrm{div}\,\tau, u) = 0,$$

thus an **integration by parts** shows that u has a weak derivative and satisfies boundary conditions, that is, u is where it should be $u \in \mathbb{H}^1_0(\Omega)$.

Mixed Methods - Galerkin Approach for Poisson

Let us build the **discrete problem**.

A property of the form

We can (but won't) show that for any partition Ω_h of Ω

$$\left\{\tau \in \mathbb{L}^2(\Omega)^n \,:\, \tau|_{\Omega_j} \in \mathbb{H}^1(\Omega_j) \text{ and } \tau|_{\Omega_j} \cdot \hat{\mathbf{n}} = \tau_{\Omega_i} \cdot \hat{\mathbf{n}} \;\forall\, \overline{\Omega}_i \cap \overline{\Omega}_j \neq \emptyset\right\} \subset \mathbb{H}^1(\Omega).$$

In layman terms, piecewise differentiable functions with continuous normal traces across elements are in $\mathbb{H}^1(\Omega)$.

This observation is crucial for building **conformal FEM spaces** for this problem.

- We could consider are the *Raviart-Thomas elements* (Raviart and Thomas 1977),
- The other usual option are the **Brezzi-Douglas-Marini elements** (Brezzi, Douglas, and Marini 1985),

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- We could consider are the *Raviart-Thomas elements* (Raviart and Thomas 1977),
- The other usual option are the **Brezzi-Douglas-Marini elements** (Brezzi, Douglas, and Marini 1985),
- To build the matrices we will use the code from (Zhang 2015).

Mixed Methods - Test problem

We consider a **more general formulation** of the Poisson problem
$\Omega \subset \mathbb{R}^2$ a polygonal domain, with boundary
$\partial\Omega = \Gamma_D \cap \Gamma_N$ $(\Gamma_D \cap \Gamma_N = \emptyset,\ \mu(\Gamma_D) \neq 0)$

$$
\begin{cases}
-\nabla \cdot (\alpha(x)\nabla u) = f, & \text{in } \Omega, \\
-\alpha \nabla u \cdot \hat{\mathbf{n}} = g_N, & \text{on } \Gamma_N, \\
u = g_D, & \text{on } \Gamma_D.
\end{cases}
$$

With

- $f \in \mathbb{L}^2(\Omega)$,
- $g_D \in \mathbb{H}^{1/2}(\Gamma_D)$ and $g_N \in \mathbb{L}^2(\Gamma_N)$,
- $\alpha(x)$ positive **piecewise constant**.

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- $\alpha(x)$ positive **piecewise constant**.

The weak form is then for $(\tau, v) \in \mathbb{H}_N(\mathrm{div}) \times \mathbb{L}^2(\Omega)$

$$\begin{cases} (\alpha^{-1}\boldsymbol{\sigma},\ \boldsymbol{\tau}) - (\mathrm{div}\,\boldsymbol{\tau},\ u) = -(\boldsymbol{\tau} \cdot \hat{\mathbf{n}}, g_D)_{\Gamma_D} \\ (\mathrm{div}\,\boldsymbol{\sigma},\ v) = (f,\ v) \end{cases}$$

where

- $\boldsymbol{\sigma} = -\alpha(x)\nabla u$ is the flux,
- $\mathbb{H}_N(\mathrm{div}) = \{\tau \in \mathbb{H}(\mathrm{div})\ : \tau \cdot \hat{\mathbf{n}} = 0 \text{ on } \Gamma_N\}$.

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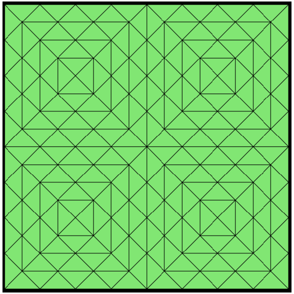
Existence theory is not substantially different, just longer to write, see (Boffi, Brezzi, and Fortin 2013, Chapter 7).

Mixed Methods - Galerkin Approach for Poisson

To apply the **discrete version of Brezzi's Theorem**, for which we select the Brezzi-Douglas-Marini and piecewise constant elements to build our mixed space.



Mesh: shape-regular affine triangulation Ω_h

- **Mesh:**

```
RefinementLevels = 2;
node = [-1 1; 0 1; 1 1; -0.5 0.5; 0.5 0.5; -1 0; 0 0; 1 0;
↪ -0.5 -0.5; 0.5 -0.5;-1.0 -1.0; 0.0 -1.0;1.0 -1.0];
elem =[4 2 1;4 1 6;4 6 7;4 7 2;5 3 2;5 2 7;5 7 8;5 8 3;9
↪  7 6; 9 6 11;9 11 12;9 12 7;10 8 7;10 7 12;10 12 13;10
↪  13 8];
bdEdge = [2 0 0;1 0 0; 0 0 0;0 0 0;2 0 0;0 0 0;0 0 0;1 0
↪  0; 0 0 0;1 0 0;1 0 0;0 0 0;0 0 0;0 0 0;1 0 0;1 0 0];
for i=1:RefinementLevels
[node,elem,bdEdge] = uniformrefine(node,elem,bdEdge);
end
```

Mixed Methods - Galerkin Approach for Poisson

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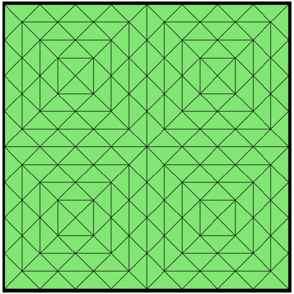
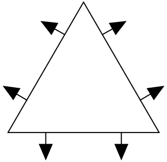


Mesh: shape-regular affine triangulation Ω_h

- **Mesh:** Ω_h
- The BDM_1 elements



In general, $\mathbf{q} \in \mathrm{BDM}_k = (\mathbb{P}_k)^2$, thus $\operatorname{div} \mathbf{q} \in \mathbb{P}_{k-1}$, and to complete the definition we impose the values on the normal trace $\phi = \mathbf{q} \cdot \hat{\mathbf{n}}$ on ∂K belonging to

$$\{\phi \,|\, \phi \in \mathbb{L}^2(K),\ \phi|_{\partial\Omega} \in \mathbb{P}_k\}.$$

Mixed Methods - Galerkin Approach for Poisson

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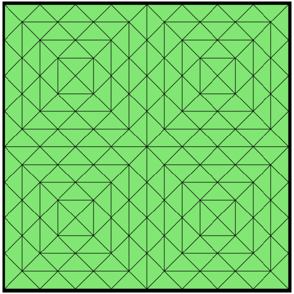

Mesh: shape-regular affine triangulation Ω_h

- **Mesh:** Ω_h
- The BDM_1 elements
- The P_0 elements

$$P_0 = \{v \,:\, v|_K \in \mathbb{P}_0(K), \quad \forall K \in \Omega_h\}.$$

Mixed Methods - Galerkin Approach for Poisson

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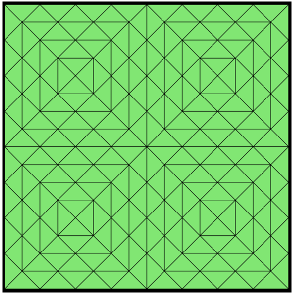

Mesh: shape-regular affine triangulation Ω_h

- **Mesh:** Ω_h
- The BDM_1 elements $\rightsquigarrow V_h$
- The P_0 elements $\rightsquigarrow M_h$
- For the convergence analysis see (Brezzi, Douglas, and Marini 1985, Section 3 and 4).

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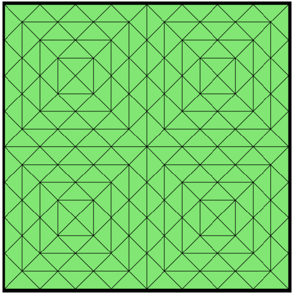


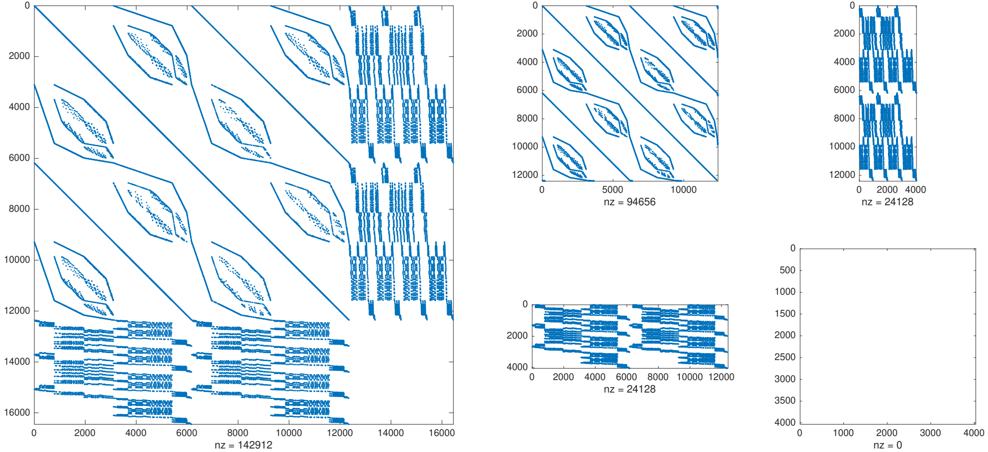
Mesh: shape-regular affine triangulation Ω_h

- **Mesh:** Ω_h
- The BDM_1 elements
- The P_0 elements
- For the convergence analysis see (Brezzi, Douglas, and Marini 1985, Section 3 and 4).
- And look at the code for assembling the matrix

```
NT = size(elem,1);          % Number of triangles
NE = size(edge,1);          % Number of edges
sol = zeros(2*NE+NT,1); % Space to store the solution
inva =1./exactalpha((node(elem(:,1)) + node(elem(:,2)) +
↪  node(elem(:,3)))/3);
[a,b,area] = gradlambda(node,elem);
M = assemblebdm(NT,NE,a,b,area,elem2edge,signedge,inva);
```

Mixed Methods - The Saddle-Point Matrix

We can finally look at our first **saddle-point** matrix for the Poisson problem.



Mixed Methods - Eigenvalue Bounds

One of the results you have seen in the **morning lectures** concerns eigenvalue bounds for these matrices. Let us look at it numerically.

Theorem (Rusten and Winther 1992)

Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0$ be the eigenvalues of A, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m > 0$ the singular values of B. If we denote as $\sigma(\mathcal{A})$ the spectrum of $\mathcal{A}$, then

$$\sigma(\mathcal{A}) \subset I = I^- \cup I^+,$$

where

$$I^- = \left[\frac{1}{2}\left(\mu_n - \sqrt{\mu_n^2 + 4\sigma_1^2}\right), \frac{1}{2}\left(\mu_1 - \sqrt{\mu_1^2 + 4\sigma_m^2}\right)\right],$$

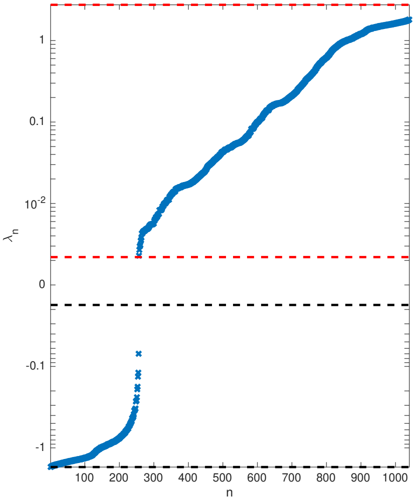
$$I^+ = \left[\mu_n, \frac{1}{2}\left(\mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2}\right)\right].$$

Mixed Methods - Eigenvalue Bounds

We can compute the bounds with few lines of code:

```
lambda = eig(M(freeDof,freeDof));
mun = eigs(A,1,'smallestabs');
mu1 = eigs(A,1,'largestabs');
sigma1 = svds(BT,1,'largest');
sigmam = svds(BT,1,'smallest');

Iminus(1) = 0.5*(mun - sqrt(mun^2+4*sigma1^2));
Iminus(2) = 0.5*(mu1 - sqrt(mu1^2+4*sigmam^2));
Iplus(1) = mun;
Iplus(2) = 0.5*(mu1 + sqrt(mu1^2 +
↪  4*sigma1^2));
```



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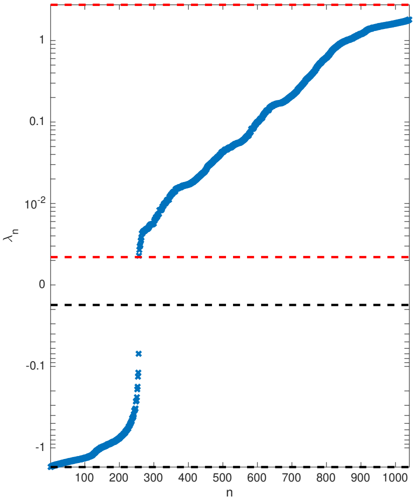
```

Next week, after you become familiar with iterative methods, we will focus on **preconditioning**.



Mixed Methods - The Stokes equation

Let us consider the Stokes equations for the *steady flow* of a **very viscous fluid**

$$\begin{cases} -\nabla^2 \mathbf{u} + \nabla p = \mathbf{0}, & \text{Momentum equation,} \\ \nabla \cdot \mathbf{u} = 0, & \text{Incompressibility constraint.} \end{cases}$$

- $\mathbf{u}$ is a *vector-valued function* representing the velocity of the fluid,
- p is a *scalar* function representing the pressure.

Modeling assumption

The crucial **modeling assumption** is that the flow is "low speed" we **neglect** effects due to **convection**.

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Let us consider the Stokes equations for the *steady flow* of a **very viscous fluid**

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Modeling assumption

The crucial **modeling assumption** is that the flow is "low speed" we **neglect** effects due to **convection**.

Why do we care?

Stokes equations represent a limiting case of the more general Navier–Stokes equations

The Stokes equation: weak formulation

Let us build the **weak formulation**

$$\begin{aligned} -\nabla^2 \mathbf{u} + \nabla p &= \mathbf{0}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned}$$

The Stokes equation: weak formulation

Let us build the **weak formulation**, we select $(\mathbf{v}, q) \in V \times M$

$$\int_\Omega \mathbf{v} \cdot \left(-\nabla^2 \mathbf{u} + \nabla p\right) = \mathbf{0},$$

$$\int_\Omega q \nabla \cdot \mathbf{u} = 0,$$

The Stokes equation: weak formulation

Let us build the **weak formulation**, we select $(\mathbf{v}, q) \in V \times M$

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} - \int_{\partial\Omega} \left(\frac{\partial \mathbf{u}}{\partial n} - p \hat{\mathbf{n}} \right) \cdot \mathbf{v} = \mathbf{0},$$

$$\int_\Omega q \nabla \cdot \mathbf{u} = 0,$$

- Here $\nabla \mathbf{u} : \nabla \mathbf{v}$ is the **componentwise** scalar product, e.g., in dimension 2, this is $\nabla u_x \cdot \nabla v_x + \nabla u_y \cdot \nabla v_y$

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- Here $\nabla \mathbf{u} : \nabla \mathbf{v}$ is the **componentwise** scalar product
- We select boundary conditions $\partial\Omega = \Gamma_N \cup \Gamma_D$ $\Gamma_D \cap \Gamma_N = \emptyset$, $\mu(\Gamma_D) \neq 0$:

$$\mathbf{u} = \mathbf{w} \text{ on } \Gamma_D, \quad \frac{\partial \mathbf{u}}{\partial n} - p\hat{\mathbf{n}} = \mathbf{s} \text{ on } \Gamma_N$$

The Stokes equation: weak formulation

Let us build the **weak formulation**, we select $(\mathbf{v}, q) \in \mathbb{H}^1_{E_0} \times \mathbb{L}^2$

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- We define the spaces

$$\mathbb{H}^1_E = \{\mathbf{u} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{u} = \mathbf{w} \text{ on } \Gamma_D\}, \quad \mathbb{H}^1_{E_0} = \{\mathbf{v} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{v} = 0 \text{ on } \Gamma_D\}.$$

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The Stokes equation: weak formulation

Let us build the **weak formulation**

$$\text{Find } (\mathbf{u}, p) \in \mathbb{H}^1_E \times \mathbb{L}^2(\Omega) \text{ s.t. } \begin{cases} \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} - \int_\Omega p\nabla\cdot\mathbf{v} - \int_{\partial\Omega} \mathbf{s}\cdot\mathbf{v} = \mathbf{0}, & \forall\, \mathbf{v} \in \mathbb{H}^1_{E_0} \\ \int_\Omega q\nabla\cdot\mathbf{u} = 0, & \forall\, q \in \mathbb{L}^2. \end{cases}$$

- Here $\nabla\mathbf{u} : \nabla\mathbf{v}$ is the **componentwise** scalar product
- We select boundary conditions $\partial\Omega = \Gamma_N \cup \Gamma_D$ $\Gamma_D \cap \Gamma_N = \emptyset$, $\mu(\Gamma_D) \neq 0$:

$$\mathbf{u} = \mathbf{w} \text{ on } \Gamma_D, \quad \frac{\partial\mathbf{u}}{\partial n} - p\hat{\mathbf{n}} = \mathbf{s} \text{ on } \Gamma_N$$

- We define the spaces

$$\mathbb{H}^1_E = \{\mathbf{u} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{u} = \mathbf{w} \text{ on } \Gamma_D\}, \quad \mathbb{H}^1_{E_0} = \{\mathbf{v} \in \mathbb{H}^1(\Omega)^d \mid \mathbf{v} = 0 \text{ on } \Gamma_D\}.$$

The Stokes equation: issues with BCs

Find $(\mathbf{u}, p) \in \mathbb{H}^1_E \times \mathbb{L}^2(\Omega)$ s.t. $\begin{cases} \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} - \int_{\partial\Omega} \mathbf{s} \cdot \mathbf{v} = \mathbf{0}, & \forall\, \mathbf{v} \in \mathbb{H}^1_{E_0}, \\ \int_\Omega q \nabla \cdot \mathbf{u} = 0, & \forall\, q \in \mathbb{L}^2. \end{cases}$

Words of caution

1. For a **unique velocity solution** the Dirichlet part of the boundary has to be nontrivial,
2. If the velocity is fixed everywhere on the boundary ($\Gamma_D \equiv \partial\Omega$) the pressure solution is only unique up to a constant (*hydrostatic pressure level*) and **w** has to satisfy

$$0 = \int_\Omega \nabla \cdot \mathbf{u} = \int_{\partial\Omega} \mathbf{u} \cdot \hat{\mathbf{n}} = \int_{\partial\Omega} \mathbf{w} \cdot \hat{\mathbf{n}},$$

i.e., the volume of fluid entering the domain must be matched by the volume of fluid flowing out of it.

The Stokes equation: Mixed Elements

As we have done for Poisson, we need to select $V_h \subset V = \mathbb{H}^1_{E_0}$ and $M_h \subset M = \mathbb{L}^2(\Omega)$:

Find $(\mathbf{u}_h, p_h) \in V_h \times M_h$ s.t.
$$\begin{cases} \int_\Omega \nabla \mathbf{u}_h : \nabla \mathbf{v}_h - \int_\Omega p_h \nabla \cdot \mathbf{v}_h - \int_{\partial\Omega} \mathbf{s} \cdot \mathbf{v}_h = \mathbf{0}, & \forall\, \mathbf{v}_h \in V_h, \\ \int_\Omega q_h \nabla \cdot \mathbf{u}_h = 0, & \forall\, q_h \in M_h. \end{cases}$$

To determine the subspaces V_h and M_h we want to apply the Theorem (Discrete Brezzi)

$$\min_{q_h \neq \text{const.}} \max_{\mathbf{v}_h \neq \mathbf{0}} \frac{|(q_h, \nabla \cdot \mathbf{v}_h)|}{\|\mathbf{v}_h\|_V \|q_h\|_M} \geq \beta.$$

where

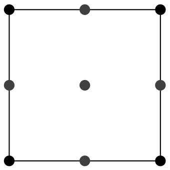
- $\|\mathbf{v}\|_V = \left(\int_\Omega \mathbf{v} \cdot \mathbf{v} + \nabla \mathbf{v} : \nabla \mathbf{v}\right)^{\frac{1}{2}}$,
- $\|q\|_M = \|q - \mu(\Omega)^{-1} \int_\Omega q\|$.

The Stokes equation: Mixed Elements

Idea for finding inf-sup stable elements

The idea is to consider "local enclosed flow Stokes problems" posed on a subdomain $\mathcal{M} \subset \Omega$ ($\mathbf{w} \cdot \hat{\mathbf{n}} = 0$ on $\partial\mathcal{M}$) called a *macroelement* that has a topology that is regular and simple enough (so that we can actually do estimates and computations).

Q_2-Q_1 Elements



Two velocity components

We approximate the 2 components of velocity with a single Q_2 FEM space

$$\begin{aligned}\{\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{2n}\} = &\{(\phi_1, 0)^T, \ldots, (\phi_n, 0)^T,\\ &(0, \phi_1)^T, \ldots, (0, \phi_n)^T\}\end{aligned}$$

The Stokes equation: Mixed Elements

Idea for finding $\inf$-$\sup$ stable elements

The idea is to consider "local enclosed flow Stokes problems" posed on a subdomain $\mathcal{M} \subset \Omega$ ($\mathbf{w} \cdot \hat{\mathbf{n}} = 0$ on $\partial\mathcal{M}$) called a *macroelement* that has a topology that is regular and simple enough (so that we can actually do estimates and computations).

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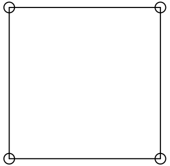

Pressure

We approximate the 2 components of velocity with a single Q_2 FEM space $\{\phi_j\}_{j=1}^{n_u}$. And the scalar pressure component with Q_1 FEM space $\{\psi_j\}_{j=1}^{n_p}$ giving:

$$\mathcal{A} = \begin{bmatrix} A & O & B_x^T \\ O & A & B_y^T \\ B_x & B_y & O \end{bmatrix} \quad \begin{aligned} a_{i,j} &= \int_\Omega \nabla\phi_i \cdot \nabla\phi_j, \\ b_{x,ki} &= -\int_\Omega \psi_k \partial_x \phi_i, \\ b_{y,kj} &= -\int_\Omega \psi_k \partial_y \phi_j. \end{aligned}$$

Since we have an enclosed flow $\ker B^T = \{\mathbf{1}\}$.

The Stokes equation: Mixed Elements

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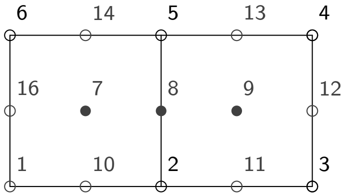
Q_2-Q_1 Macroelement



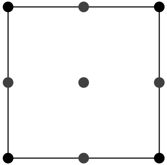
Three interior velocity nodes and six pressure nodes.

B^T is a 6×6 matrix, with some effort we can compute all the entries and verify that $\ker B^T = \{\mathbf{1}\}$ (part of the computations are done in (Elman, Silvester, and Wathen 2014, Section 3.3.1)).

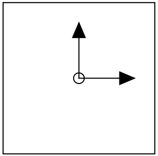
- Then stability holds for all patches of elements with the same topology,
- Any grid made of of an even number of cell can be decomposed this way.

The Stokes equation: other stable elements

Q_2-P_{-1} Elements

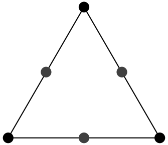


Velocity



Pressure, and pressure derivatives

P_2-P_1 Elements



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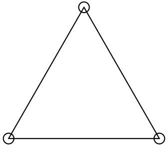
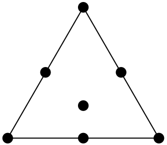
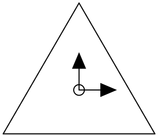


Pressure

P_{2*}-P_{-1} Element



Velocity

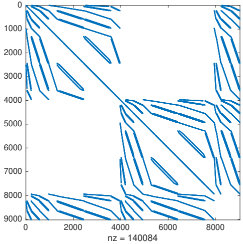


Pressure, and pressure derivatives

The Stokes equation: the associated saddles

The P_2-P_1 (Taylor-Hood) case for the **colliding flow** test problem.

- $\Omega = [-1,1] \times [-1,1]$
- $u_x = 20xy^3$, $u_y = 5x^4 - 5y^4$, $p = 60x^2y - 20y^3 + c$,
- Dirichlet boundary condition on all the square $\psi(x,y) = 5xy^4 - x^5$.




```
%% Building the mesh
RefinementLevels = 2;
square = [0,1,0,1];
h = 0.25;
[node,elem] = squaremesh(square,h);
for i=1:RefinementLevels
 [node,elem] = uniformrefine(node,elem);
end
%% Building the test problem: colliding flows
bdFlag = setboundary(node,elem,'Dirichlet');
pde = Stokesdata1;
options.solver='none';  % We just perform the build
[soln,eqn,info] =
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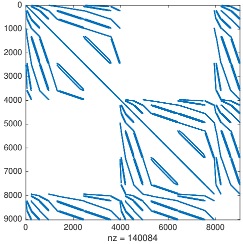


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Question:

Can we apply (Rusten and Winther 1992)?

The Stokes equation: properties of the matrix

Theorem (Elman, Silvester, and Wathen 2014, Theorem 3.2.1)

With P_1, P_2, Q_1 or Q_2 approximation on a *shape-regular*, *quasi-uniform* subdivision of $\mathbb{R}^2$, the matrix A for the *discrete vector Laplacian* satisfies

$$ch^2 \leq \frac{< A\mathbf{v}, \mathbf{v} >}{< \mathbf{v}, \mathbf{v} >} \leq C \qquad \forall\, \mathbf{v} \in \mathbb{R}^{n_u},$$

where h is the length of the longest edge in the mesh, and c and C are **constants independent** of h.

- This gives us information the behavior of the **smallest** and **largest eigenvalue** of A (*Rayleigh Principle*)!

h^2	$\lambda_{\min}(A)$	$h^2/\lambda_{\min}(A)$	$\lambda_{\max}(A)$
0.0156	0.0768	0.2035	10.5391
0.0039	0.0193	0.2028	10.6346
0.0010	0.0048	0.2027	10.6586
0.0002	0.0012	0.2027	10.6647

The Stokes equation: properties of the matrix

To uncover information on the B matrices, we need to introduce a discrete representation of the norm of $M_h \subset \mathbb{L}^2$:

$$p_h \in M_h : \|p_h\| =< Qp_h, p_h >^{1/2}, \quad Q = (q_{kl}), \; q_{k,l} = \int_\Omega \psi_k \psi_l, \; k, l = 1, \ldots, n_p.$$

! The matrix Q is called **mass matrix** for the pressure space, *in general*, we call mass-matrices all the matrices obtained in this way for the basis of a given FEM space.

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Generalized singular values

We call **generalized singular values** the **real** numbers σ associated with the following generalized eigenvalue problem

$$\begin{bmatrix} O & B^T \\ B & O \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{q} \end{bmatrix} = \sigma \begin{bmatrix} A & O \\ O & Q \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{q} \end{bmatrix}.$$

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$\sigma = 0$ this implies $B^T\mathbf{q} = 0$, and $B\mathbf{v} = 0$,

$\sigma \neq 0$ we select vector $(\mathbf{v}^T, -\mathbf{q}^T)^T$ and obtain

$$< \mathbf{v}, B^T\mathbf{q} > - < \mathbf{q}, B\mathbf{v} > = 0 = \sigma\left(< \mathbf{v}, A\mathbf{v} > - < \mathbf{q}, Q\mathbf{q} >\right)$$
$$\Rightarrow < A\mathbf{v}, \mathbf{v} > = < Q\mathbf{q}, \mathbf{q} > .$$

That is

$$\frac{< BA^{-1}B^T\mathbf{q}, \mathbf{q} >}{< Q\mathbf{q}, \mathbf{q} >} = \sigma^2 = \frac{< B^TQ^{-1}B\mathbf{v}, \mathbf{v} >}{< A\mathbf{v}, \mathbf{v} >}.$$

The Stokes equation: properties of the matrix

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- If $\ker B^T = \mathbf{0}$ then B has n_p positive singular values.
- This is linked to the $\inf\text{-}\sup$ condition:

$$\beta \le \min_{q_h \neq \text{const}} \max_{\mathbf{v}_h \neq \mathbf{0}} \frac{|(q_h, \nabla\cdot\mathbf{v}_h)|}{\|\nabla\mathbf{v}_h\|\|\mathbf{q}_h\|} = \min_{\mathbf{q}\neq\mathbf{1}} \max_{\mathbf{v}\neq 0} \frac{|< \mathbf{q}, B\mathbf{v} >|}{< A\mathbf{v},\mathbf{v} >^{1/2}< Q\mathbf{q},\mathbf{q} >^{1/2}}$$

$$= \min_{\mathbf{q}\neq\mathbf{1}} \frac{1}{< Q\mathbf{q},\mathbf{q} >^{1/2}} \max_{\mathbf{w}=A^{1/2}\mathbf{v}\neq\mathbf{0}} \frac{|< \mathbf{q}, BA^{-1/2}\mathbf{w} >|}{< \mathbf{w},\mathbf{w} >^{1/2}}$$

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The Stokes equation: properties of the matrix

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$$= \min_{\mathbf{q} \neq \mathbf{1}} \frac{1}{< Q\mathbf{q}, \mathbf{q} >^{1/2}} \max_{\mathbf{w} = A^{1/2}\mathbf{v} \neq \mathbf{0}} \frac{| < A^{-1/2} B^T \mathbf{q}, \mathbf{w} > |}{< \mathbf{w}, \mathbf{w} >^{1/2}}$$

$$= \min_{\mathbf{q} \neq \mathbf{1}} \frac{< A^{-1/2} B^T \mathbf{q}, A^{-1/2} B^T \mathbf{q} >^{1/2}}{< Q\mathbf{q}, \mathbf{q} >^{1/2}} = \min_{\mathbf{q} \neq \mathbf{1}} \frac{< BA^{-1}B^T \mathbf{q}, \mathbf{q} >^{1/2}}{< Q\mathbf{q}, \mathbf{q} >^{1/2}} = \sigma_{\min}$$

The Stokes equation: properties of the matrix

Theorem (Elman, Silvester, and Wathen 2014, Theorem 3.22)

Let $\partial\Omega \equiv \Gamma_D$, the Stokes problem discretized with a uniformly stable mixed approximation on a shape-regular, quasi-uniform subdivision of $\mathbb{R}^2$, has a **Schur complement matrix** $BA^{-1}B^T$ that is spectrally equivalent to the pressure mass matrix Q:

$$\beta^2 \leq \frac{< BA^{-1}B^T\mathbf{q}, \mathbf{q} >}{< Q\mathbf{q}, \mathbf{q} >} \leq 1, \ \forall \, \mathbf{q} \in \mathbb{R}^{n_p} \ : \ \mathbf{q} \neq \mathbf{1}.$$

The $\inf$-$\sup$ constant β is bounded away from zero independently of h and the condition number (discarding the zero eigenvalue) $\kappa^e(BA^{-1}B^T) \leq C/(c\beta)^2$ for c and C given by

$$ch^2 \leq \frac{< Q\mathbf{q}, \mathbf{q} >}{< \mathbf{q}, \mathbf{q} >} \leq Ch^2, \quad \forall \, \mathbf{q} \in \mathbb{R}^{n_p}.$$

The Stokes equation: properties of the matrix

We can **run this test** on the usual test problem by running the code in the folder

</> `E3-Stokes/stokesmatrixproperties.m`

This tests both:

1. The bound on the vector Laplacian,
2. The bounds on the Schur complement.

for the P_2-P_1 elements.

h	λ_2	λ_{n_p}
0.2500	0.1352	0.9932
0.1250	0.1341	0.9996
0.0625	0.1336	1.0000
0.0312	0.1334	1.0000

Generalized eigenvalues for:

$$BA^{-1}B^T\mathbf{x} = \lambda Q\mathbf{x}$$

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Why do we care?

As you will see in the following, these information are **useful** for the **design of iterative solvers**.

The Stokes equation: stabilized discretizations

We have seen that the matter of obtain a **stable discretization** depends on the null-space of B^T.

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Idea behind stabilization

If the discretization is not stable $\exists \mathbf{p} \neq \mathbf{1}$ such that $B^T\mathbf{p} = \mathbf{0}$, that is $(\mathbf{0}^T, \mathbf{p}^T)^T$ is a null vector for the homogeneous saddle-point system. The idea behind stabilization is **relaxing the incompressibility constraint** so that this vector is no longer in the kernel **and** we still obtain a reasonable error bound for the convergence of the method.

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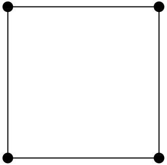
Technique

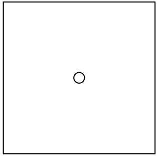
The **technique** to devise stabilization is again using *macroelements*.

The Stokes equation: stabilized Q_1-P_0

- This is the simplest *unstable* element,

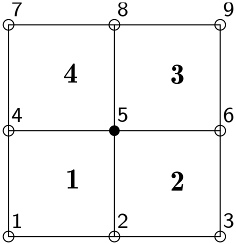


Velocity



Pressure

The Stokes equation: stabilized Q_1-P_0



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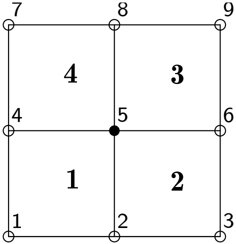

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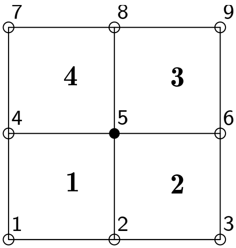
- To devise a stabilization we need a 2×2 *macroelement*
- The pressure coefficient $\mathbf{p}$ solves

$$BA^{-1}B^T\mathbf{p} = BA^{-1}\mathbf{f} - \mathbf{g},$$

the Schur complement $BA^{-1}B^T$ has **two eigenvectors** for the 0 eigenvalue: $\mathbf{1}$, and $\mathbf{q}_2 = (1, -1, 1, -1)^T$.

The Stokes equation: stabilized Q_1-P_0



$$\begin{bmatrix} A & B^T \\ B & -\gamma C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

- This is the simplest *unstable* element,
- To devise a stabilization we need a 2×2 *macroelement*
- The pressure coefficient $\mathbf{p}$ solves

$$BA^{-1}B^T\mathbf{p} = BA^{-1}\mathbf{f} - \mathbf{g},$$

  the Schur complement $BA^{-1}B^T$ has **two eigenvectors** for the 0 eigenvalue: $\mathbf{1}$, and $\mathbf{q}_2 = (1, -1, 1, -1)^T$.
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$$S_\gamma \mathbf{p} \equiv (BA^{-1}B^T + \gamma C)\mathbf{p} = BA^{-1}\mathbf{f} - \mathbf{g},$$

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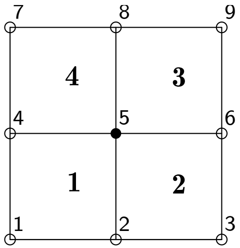

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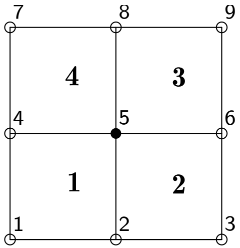

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- What about γ?

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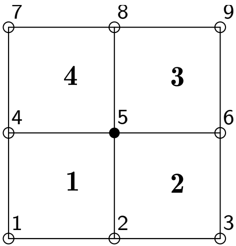

- We have moved the 0 eigenvalue with eigenvector $\mathbf{q}_2$ to 4γ, if we take $\gamma = 1/4$ we get a spectral orthogonal projector for C,

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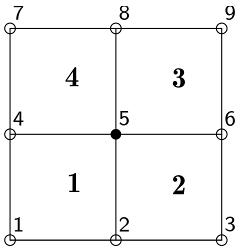

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- But γ has to be selected to balance both **stability** and **accuracy**, a natural choice would be selecting

$$\gamma_* = \frac{1}{4} h_x h_y, \qquad h_x h_y = \text{“area of the element”}$$

If Q is the **mass matrix** associate with the P_0 elements ($Q = h_x h_y \times I$), then γ_* is the largest value for which

$$\gamma \frac{\mathbf{p}^T C_* \mathbf{p}}{\mathbf{p}^T Q \mathbf{p}} \leq 1 \; \forall \, \mathbf{p} \in \mathbb{R}^{n_p}.$$

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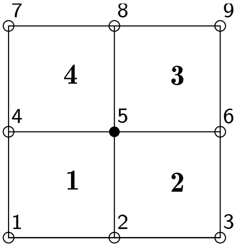

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- Complete stabilization matrix: $C = \operatorname{blockdiag}(C_*, \ldots, C_*)$.

The Stokes equation: stabilized Q_1-P_0

Other stabilization are possible

This is not the only possible stabilization matrix, other choices are possible, consider, e.g.,

$$C^* = h_x h_y \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

that has again the same eigenvectors of S_0, and is called the jump stabilization matrix.

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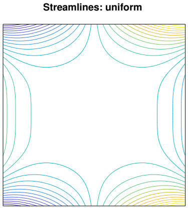
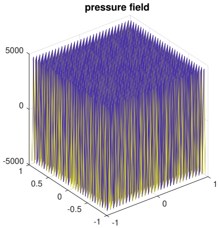
We can visually see the effect of the stabilization on the same **colliding flow** problem.

$$\gamma = 0$$





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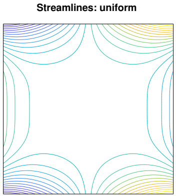
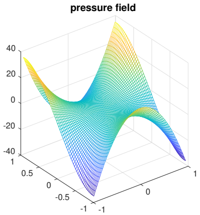
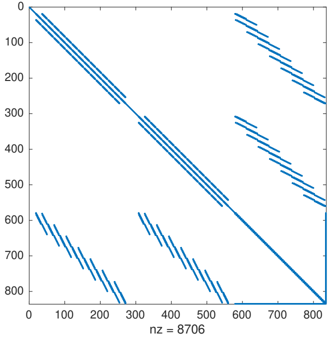




The Stokes equation: stabilized matrix properties

What can we say about the **spectral properties** of the **stabilized matrices**?



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- To substitute the $\inf\text{-}\sup$ condition we introduce the operator:

  $s(q_h) : M_h \to \mathbb{R}$

  $$q_h \mapsto s(q_h) = \max_{\mathbf{v}_h \neq \mathbf{0}} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|} + c(q_h, q_h)^{1/2},$$

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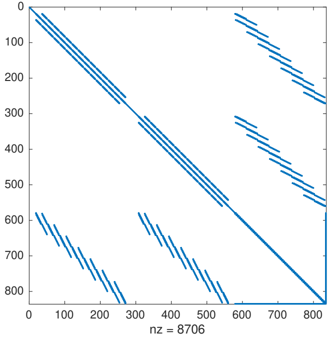

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Uniform stabilization

The Stokes problem is said to be **uniformly stabilized** if there exists β **indepedent of** h such that

$$s(q_h) \geq \beta^2 \|q_h\|, \ \forall q_h \in M_h.$$

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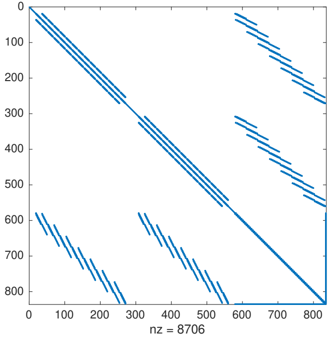

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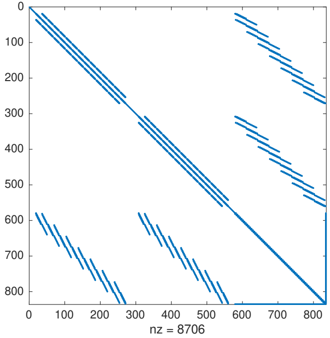

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- Then the generalized inf-sup conditions is

$$\beta^2 = 2 \min_{\mathbf{q} \neq \mathbf{1}} \frac{< (BA^{-1}B^T + C)\mathbf{q}, \mathbf{q} >}{< Q\mathbf{q}, \mathbf{q} >}.$$

The Stokes equation: stabilized matrix properties

Theorem (Elman, Silvester, and Wathen 2014, Theorem 3.29)

Let $\partial\Omega \equiv \Gamma_D$, the Stokes problem discretized with an *ideally* stabilized mixed approximation on a shape-regular, quasi-uniform subdivision of $\mathbb{R}^2$, has a **Schur complement matrix** $BA^{-1}B^T + C$ that is spectrally equivalent to the pressure mass matrix Q:

$$\beta^2 \leq \frac{< (BA^{-1}B^T + C)\mathbf{q}, \mathbf{q} >}{< Q\mathbf{q}, \mathbf{q} >} \leq 2, \; \forall \mathbf{q} \in \mathbb{R}^{n_p} \; : \; \mathbf{q} \neq \mathbf{1}.$$

The *generalized* inf-sup constant β is bounded away from zero independently of h.

$\ell(h = 2^{-\ell})$	β^2	λ_{n_p}
3	0.280929	1.7238
4	0.252201	1.74406
5	0.233876	1.74859
6	0.221837	1.74965

The **colliding flow** problem can be tested with

</> `E3-Stokes/stokesmatrixpropertiesstab.m`

The Navier-Stokes Equation

We add to the Stokes problem a **forcing term** and a **convection term** obtaining

$$\begin{aligned} -\nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \end{aligned}$$

where

- $\nu > 0$ is the *kinematic viscosity*,
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$\partial\Omega \equiv \Gamma_D$

If the velocity is specified everywhere on the boundary, then the pressure solution to the Navier–Stokes problem is only unique up to a *hydrostatic constant*.

The Navier-Stokes Equation: normalization

We normalize the system

$$\begin{aligned} -\nu\nabla^2\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla\cdot\mathbf{u} &= 0, \end{aligned}$$

to better highlight if the system is **diffusion dominated** or **advection dominated**.

- Let L denote a *characteristic length scale* for the domain Ω,
- we scale space variables as $\boldsymbol{\xi} = {}^{\mathbf{x}}/_{L}$.

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$$-\frac{1}{\mathcal{R}}\nabla^2\mathbf{u}_* + \mathbf{u}_*\cdot\nabla\mathbf{u}_* + \nabla p_* = \frac{L}{U^2}\mathbf{f}, \quad \mathcal{R} = {}^{UL}/_{\nu}$$
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Reynolds number

We call $\mathcal{R} = {}^{UL}/_{\nu}$ the Reynolds number. If $\mathcal{R} \leq 1$ then the problem is diffusion dominated, for increasing values of $\mathcal{R}$ we get instead convection dominated problems.

The Navier-Stokes Equation: weak formulation

Can be written *similarly to the Stokes problem*

$$\text{Find } (\mathbf{u}, p) \in V \times M \ : \begin{cases} \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} + \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} - \int_\Omega p(\nabla \cdot \mathbf{v}) = \int_\Omega \mathbf{f} \cdot v, \\ \int_\Omega q(\nabla \cdot \mathbf{u}) = 0. \end{cases}$$

We need again the **suitable spaces**

- $\mathbf{u} \in \mathbb{H}^1_E = \{\mathbf{u} \in \mathbb{H}^1(\Omega)^d \,|\, \mathbf{u} = \mathbf{w} \text{ on } \Gamma_D\} \equiv V$, $p \in \mathbb{L}^2(\Omega) \equiv M$,
- $\mathbf{v} \in \mathbb{H}^1_{E_0} = \{\mathbf{v} \in \mathbb{H}^1(\Omega)^d \,|\, \mathbf{v} = 0 \text{ on } \Gamma_D\}$,
- New addition is a **trilinear form** for the **velocity term**:

$$c \ : \mathbb{H}^1_{E_0} \times \mathbb{H}^1_{E_0} \times \mathbb{H}^1_{E_0} \to \mathbb{R}$$

$$(\mathbf{z}, \mathbf{u}, \mathbf{v}) \mapsto c(\mathbf{z}, \mathbf{u}, \mathbf{v}) = \int_\Omega (\mathbf{z} \cdot \nabla \mathbf{u}) \cdot \mathbf{v}.$$

The Navier-Stokes Equation: existence

This is a **non linear** problem, so for existence we need both *Lax-Milgram* and a result for *nonlinear systems of algebraic equations*.

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- We restate the problem

$$\text{Find } (u, p) \in V \times M : \begin{cases} a(\mathbf{u}, v) + c(\mathbf{u}, \mathbf{u}, v) + b(\mathbf{v}, p) = (\mathbf{f}, v), & \forall\, \mathbf{v} \in V, \\ b(\mathbf{u}, q) = 0, & \forall, q \in M, \end{cases} \quad \text{(NS)}$$

with

$$a(\mathbf{w}, \mathbf{v}) = \nu(\nabla\mathbf{w}, \nabla\mathbf{v}) : V \times V \to \mathbb{R},\ b : V \times Q \to \mathbb{R} = -(q, \nabla \cdot \mathbf{v}),$$

$$c(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^{d} \left(u_j \frac{\partial u_i}{\partial x_j}, v_i \right) : V \times V \times V \to \mathbb{R}$$

The Navier-Stokes Equation: existence

The existence proof then follows in few steps.

1. We consider the problem on the space $V_{\mathrm{div}} = \{\mathbf{v} \in \mathbb{H}^1(\Omega)^d \,:\, \mathrm{div}\,\mathbf{v} = 0\}$, then a solution of (NS) is a solution also of the problem on this space

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2. Then we prove that to a solution on the reduced space corresponds a solution of the full problem.

Lemma (Quarteroni and Valli 1994, Lemma 10.1.1)

Let $\mathbf{u}$ be a solution of (NS$_{\text{div}}$). Then there exists a unique $p \in M$ such that $(\mathbf{u}, p)$ is a solution of problem (NS).

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3. Prove that $(\mathrm{NS}_{\mathrm{div}})$ has a *unique* solution.

The Navier-Stokes Equation: existence

Theorem (Quarteroni and Valli 1994, Theorem 10.1.1)

Let $\mathbf{f} \in \mathbb{H}_{\mathrm{div}} = \{\mathbf{v} \in \mathbb{L}^2(\Omega)^d \mid \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} \cdot \hat{\mathbf{n}} = 0 \text{ on } \partial\Omega\}$, with

$$\|\mathbf{f}\| < \frac{\nu^2}{\hat{C} C_\Omega^{1/2}},$$

where $\hat{C} > 0$ is the *continuity constant* for the trilinear form c, i.e.,

$$|c(\mathbf{w}, \mathbf{z}, \mathbf{v})| \leq \hat{C} |\mathbf{w}|_1 |\mathbf{z}|_1 |\mathbf{v}|_1 \quad \forall\, \mathbf{w}, \mathbf{z}, \mathbf{v} \in \mathbb{H}_0^1(\Omega)^d,$$

and C_Ω is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\mathrm{div}}$ to ($\mathrm{NS}_{\mathrm{div}}$).

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and C_Ω is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\mathrm{div}}$ to ($\mathrm{NS}_{\mathrm{div}}$).

Idea of the proof.

1. Use Lax-Milgram for problem $\mathcal{A}_{\mathbf{w}}(\mathbf{z}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$, $\forall, \mathbf{v} \in V_{\mathrm{div}}$ and $\mathcal{A}_{\mathbf{w}}(\mathbf{z}, \mathbf{v}) = a(\mathbf{z}, \mathbf{v}) + c(\mathbf{w}, \mathbf{z}, \mathbf{v})$ to prove existence for every $\mathbf{w}$.

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and C_Ω is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\mathrm{div}}$ to ($\mathrm{NS}_{\mathrm{div}}$).

Idea of the proof.

2. The solution we look for is then a fixed point of the map $\Phi : \mathbf{w} \to \mathbf{z}$. First we prove that such solution is in a ball in V_{div}.

The Navier-Stokes Equation: existence

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$$|c(\mathbf{w}, \mathbf{z}, \mathbf{v})| \leq \hat{C} |\mathbf{w}|_1 |\mathbf{z}|_1 |\mathbf{v}|_1 \quad \forall\, \mathbf{w}, \mathbf{z}, \mathbf{v} \in \mathbb{H}_0^1(\Omega)^d,$$

and C_Ω is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\mathrm{div}}$ to $(\mathrm{NS}_{\mathrm{div}})$.

Idea of the proof.

3. Finally, we apply *Banach contraction Theorem* (using the hypothesis on $\mathbf{f}$) to prove that there exist a unique fixed point for the problem.

The Navier-Stokes Equation: existence

Theorem (Quarteroni and Valli 1994, Theorem 10.1.1)

Let $\mathbf{f} \in \mathbb{H}_{\mathrm{div}} = \{\mathbf{v} \in \mathbb{L}^2(\Omega)^d \mid \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}\cdot\hat{\mathbf{n}} = 0 \text{ on } \partial\Omega\}$, with

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where $\hat{C} > 0$ is the *continuity constant* for the trilinear form c and C_Ω is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\mathrm{div}}$ to ($\mathrm{NS}_{\mathrm{div}}$).

Conditions - 1

It is not restrictive to assume $\mathbf{f} \in \mathbb{H}_{\mathrm{div}}$, any $\mathbf{f} \in \mathbb{L}^2(\Omega)^d$ can be decomposed as the sum of a function in $\mathbb{H}_{\mathrm{div}}$ and a function that is a gradient of an $\mathbb{H}^1(\Omega)$ function. The gradient component of the external force field $\mathbf{f}$ doesn't play a role, $(\mathbf{v}, \nabla q) = 0\ \forall\, q \in \mathbb{H}^1$ and $\mathbf{v} \in \mathbb{H}_{\mathrm{div}}$.

The Navier-Stokes Equation: existence

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Let $\mathbf{f} \in \mathbb{H}_{\mathrm{div}} = \{\mathbf{v} \in \mathbb{L}^2(\Omega)^d \mid \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} \cdot \hat{\mathbf{n}} = 0 \text{ on } \partial\Omega\}$, with

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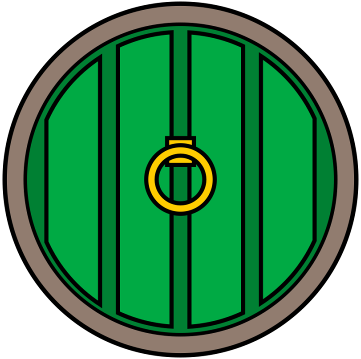
where $\widehat{C} > 0$ is the *continuity constant* for the trilinear form c and C_Ω is Poincaré constant for the domain under consideration. Then, there exist a unique solution $\mathbf{u} \in V_{\mathrm{div}}$ to $(\mathrm{NS}_{\mathrm{div}})$.

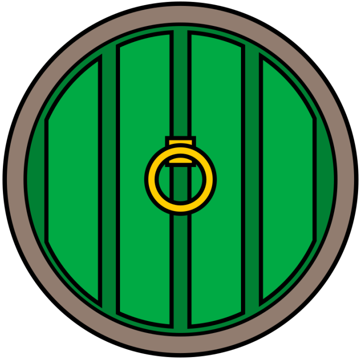
Conditions - 2

The smallness condition on the viscosity ν is necessary for proving uniqueness, and is restrictive. The solution may not be unique when ν is small w.r.t. $\mathbf{f}$, even for reasonable $\mathbf{f}$.

The Navier-Stokes Equation: linearizations

Since we only now how to solve linear problems, to face (NS) we discuss two types of **nonlinear iteration** with a linearized problem being solved at every step.





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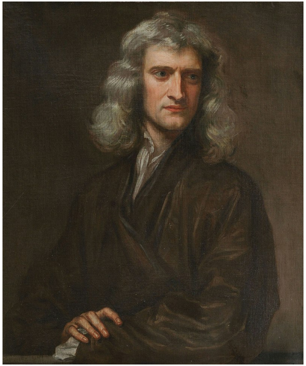

Newton Method

Picard's Iteration

The Navier-Stokes Equation: linearizations

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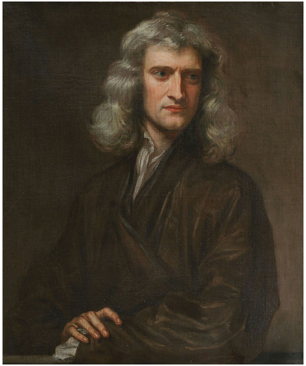

Newton Method

Picard's Iteration

We introduce both method first in the continuous context.

The Navier-Stokes Equation: Newton method

1. We have a guess $\{\mathbf{u}_k, p_k\}$ for the solution,
2. We compute the residual pairs

$$\begin{bmatrix} R_k \\ r_k \end{bmatrix} = \begin{bmatrix} \int_\Omega \mathbf{f} \cdot \mathbf{v} - c(\mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) - \nu \int_\Omega \nabla \mathbf{u}_k : \nabla \mathbf{v} + \int_\Omega p_k (\nabla \cdot \mathbf{v}) \\ - \int_\Omega q (\nabla \cdot \mathbf{u}_k) \end{bmatrix} \quad \begin{array}{l} \mathbf{v} \in \mathbb{H}^1_{E_0}, \\ q \in \mathbb{L}^2(\Omega). \end{array}$$

3. Then update the solution as

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \delta \mathbf{u}_k, \quad p_{k+1} = p_k + \delta p_k,$$

for $\delta \mathbf{u}_k \in \mathbb{H}^1_{E_0}$ and $\delta p_k \in \mathbb{L}^2(\Omega)$ the solution of

$$\begin{cases} c(\delta \mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) + c(\mathbf{u}_k, \delta \mathbf{u}_k, \mathbf{v}) + \nu \int_\Omega \nabla \delta \mathbf{u}_k : \nabla \mathbf{v} - \int_\Omega \delta p_k (\nabla \cdot \mathbf{v}) = R_k, & \forall \mathbf{v} \in \mathbb{H}^1_{E_0}, \\ \int_\Omega q (\nabla \cdot \delta \mathbf{u}_k) = r_k, & \forall\, q \in \mathbb{L}^2 \end{cases}$$

The Navier-Stokes Equation: Discrete Newton

As we have done for the **Stokes problem** we select $V_h \subset \mathbb{H}^1_{E_0}$ and $M_h \subset \mathbb{L}^2(\Omega)$,

- The Newton **updates** are then computed by solving $\forall \mathbf{v} \in V_h$, $\forall q_h \in M_h$

$$\begin{cases} c(\delta\mathbf{u}_h^{(k)}, \mathbf{u}_h^{(k)}, \mathbf{v}_h) + c(\mathbf{u}_h^{(k)}, \delta\mathbf{u}_h^{(k)}, \mathbf{v}) + \nu \int_\Omega \nabla \delta\mathbf{u}_h^{(k)} : \nabla \mathbf{v}_h - \int_\Omega \delta p_h^{(k)} (\nabla \cdot \mathbf{v}_h) = R_h^{(k)} \\ \int_\Omega q_h (\nabla \cdot \delta\mathbf{u}_h^{(k)}) = r_h^{(k)} \end{cases}$$

  where $R_k(\mathbf{v}_h)$, and $r_k(q_h)$ are the nonlinear residuals w.r.t. discrete formulation.

- Selecting basis $V_h = \mathrm{Span}\{\phi_j\}$, $M_h = \mathrm{Span}\{\psi_j\}$ and representing (dropping the k)

$$\mathbf{u}_h = \sum_{j=1}^{n_u} \mathbf{u}_j \phi_j + \sum_{n_u+1}^{n_u+n_\partial} \mathbf{u}_j \phi_j, \quad p_h = \sum_{k=1}^{n_p} \mathbf{p}_k \psi_k,$$

  and

$$\delta\mathbf{u}_h \sum_{j=1}^{n_u} + \boldsymbol{\delta}\mathbf{u}_j \phi_j, \qquad \delta p_h = \sum_{k=1}^{n_p} \boldsymbol{\delta}\mathbf{p}_k \psi_k,$$

  we get the corresponding discrete system

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$$\begin{cases} c(\delta \mathbf{u}_h^{(k)}, \mathbf{u}_h^{(k)}, \mathbf{v}_h) + c(\mathbf{u}_h^{(k)}, \delta \mathbf{u}_h^{(k)}, \mathbf{v}) + \nu \int_\Omega \nabla \delta \mathbf{u}_h^{(k)} : \nabla \mathbf{v}_h - \int_\Omega \delta p_h^{(k)} (\nabla \cdot \mathbf{v}_h) = R_h^{(k)} \\ \int_\Omega q_h (\nabla \cdot \delta \mathbf{u}_h^{(k)}) = r_h^{(k)} \end{cases}$$

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- we get the corresponding discrete system

$$\mathcal{A}\delta = \begin{bmatrix} \nu \mathbf{A} + N + \mathbf{W} & B^T \\ B & O \end{bmatrix} \begin{bmatrix} \delta \mathbf{u} \\ \delta \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

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$$\mathbf{N} = \begin{bmatrix} N & O \\ O & N \end{bmatrix} \qquad \mathbf{n}_{i,j} = \int_\Omega (\mathbf{u}_h \cdot \nabla \phi_j) \cdot \phi_i,$$

$$\mathbf{W} = \begin{bmatrix} W_{xx} & W_{xy} \\ W_{yx} & W_{yy} \end{bmatrix} \qquad \mathbf{w}_{i,j} = \int_\Omega (\phi_j \cdot \nabla \mathbf{u}_h) \cdot \phi_i,$$

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- If we use *unstable* elements, we need a stabilization matrix.

The Navier-Stokes Equation: Picard's Iteration

The second approach for linearization is **Picard's iteration**, we start again from

1. We have a guess $\{\mathbf{u}_k, p_k\}$ for the solution,
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$$\begin{bmatrix} R_k \\ r_k \end{bmatrix} = \begin{bmatrix} \int_\Omega \mathbf{f}\cdot\mathbf{v} - c(\mathbf{u}_k,\mathbf{u}_k,\mathbf{v}) - \nu\int_\Omega \nabla\mathbf{u}_k : \nabla\mathbf{v} + \int_\Omega p_k(\nabla\cdot\mathbf{v}) \\ -\int_\Omega q(\nabla\cdot\mathbf{u}_k) \end{bmatrix} \quad \begin{array}{l} \mathbf{v}\in\mathbb{H}^1_{E_0}, \\ q\in\mathbb{L}^2(\Omega). \end{array}$$

3. Then update the solution as

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \delta\mathbf{u}_k, \quad p_{k+1} = p_k + \delta p_k,$$

for $\delta\mathbf{u}_k \in \mathbb{H}^1_{E_0}$ and $\delta p_k \in \mathbb{L}^2(\Omega)$ the solution of

$$\begin{cases} c(\delta\mathbf{u}_k,\mathbf{u}_k,\mathbf{v}) + c(\mathbf{u}_k,\delta\mathbf{u}_k,\mathbf{v}) + \nu\int_\Omega \nabla\delta\mathbf{u}_k : \nabla\mathbf{v} - \int_\Omega \delta p_k(\nabla\cdot\mathbf{v}) = R_k, & \forall\mathbf{v}\in\mathbb{H}^1_{E_0}, \\ \int_\Omega q(\nabla\cdot\delta\mathbf{u}_k) = r_k, & \forall\, q\in\mathbb{L}^2 \end{cases}$$

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for $\delta \mathbf{u}_k \in \mathbb{H}^1_{E_0}$ and $\delta p_k \in \mathbb{L}^2(\Omega)$ the solution of the Oseen system

$$\begin{cases} c(\mathbf{u}_k, \delta \mathbf{u}_k, \mathbf{v}) + \nu \int_\Omega \nabla \delta \mathbf{u}_k : \nabla \mathbf{v} - \int_\Omega \delta p_k (\nabla \cdot \mathbf{v}) = R_k, & \forall \mathbf{v} \in \mathbb{H}^1_{E_0}, \\ \int_\Omega q (\nabla \cdot \delta \mathbf{u}_k) = r_k, & \forall\, q \in \mathbb{L}^2 \end{cases}$$

The Navier-Stokes Equation: Discrete Picard

The **discrete system** is the same of the Newton method **without** the Newton matrix $\mathbf{W}$:

$$\mathcal{A}\delta = \begin{bmatrix} \nu\mathbf{A} + \mathbf{N} & B^T \\ B & O \end{bmatrix} \begin{bmatrix} \delta\mathbf{u} \\ \delta\mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

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The **discrete system** is the same of the Newton method **without** the Newton matrix $\mathbf{W}$:

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- If we use *unstable* elements, we need a stabilization matrix.

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- If we use *unstable* elements, we need a stabilization matrix.

Theorem

Consider the generic saddle-point system

$$\mathcal{A} = \begin{bmatrix} \mathbf{F} & B^T \\ B & -C \end{bmatrix},$$

where C is symmetric and positive-semidefinite matrix. If $< \mathbf{F}\mathbf{u}, \mathbf{u} > 0 \ \forall \mathbf{u} \neq \mathbf{0}$, then

$$\ker \mathcal{A} = \left\{ \begin{bmatrix} \mathbf{0} \\ \mathbf{p} \end{bmatrix} \middle| .\mathbf{p} \in \ker(B\mathbf{F}^{-1}B^T + C) \right\}.$$

The Navier-Stokes Equation: Newton and Picard

Newton

$$\mathcal{A} = \begin{bmatrix} \nu A + N + W_{xx} & W_{xy} & B_x^T \\ W_{yx} & \nu A + N + W_{yy} & B_y^T \\ B_x & B_y & O \end{bmatrix}$$

- Coupled $\mathcal{A}_{1,1}$ block,
- Quadratic convergence,
- Locally convergent for "large enough" ν, and "close enough" initial guess.

Picard

$$\mathcal{A} = \begin{bmatrix} \nu A + N & O & B_x^T \\ O & \nu A + N & B_y^T \\ B_x & B_y & O \end{bmatrix}$$

- Decoupled $\mathcal{A}_{1,1}$ block,
- Linear convergence,
- Converges under the existence condition: $\|\mathbf{f}\| < \nu^2/\hat{c}\, c_\Omega^{1/2}$.

The Navier-Stokes Equation: Newton and Picard

Newton

$$\mathcal{A} = \begin{bmatrix} \nu A + N + W_{xx} & W_{xy} & B_x^T \\ W_{yx} & \nu A + N + W_{yy} & B_y^T \\ B_x & B_y & O \end{bmatrix}$$

- Coupled $\mathcal{A}_{1,1}$ block,
- Quadratic convergence,
- Locally convergent for "large enough" ν, and "close enough" initial guess.

Picard

$$\mathcal{A} = \begin{bmatrix} \nu A + N & O & B_x^T \\ O & \nu A + N & B_y^T \\ B_x & B_y & O \end{bmatrix}$$

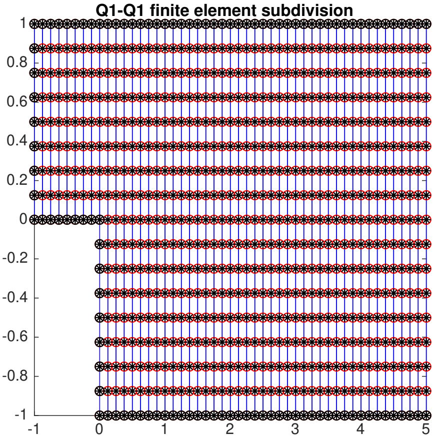
- Decoupled $\mathcal{A}_{1,1}$ block,
- Linear convergence,
- Converges under the existence condition: $\|\mathbf{f}\| < \nu^2 / \hat{c}\, c_\Omega^{1/2}$.

Next week we will delve into some **numerical experiments**, and try several **preconditioners** discussed in the morning lectures.

Navier-Stokes: backward facing step

Test problem:

- L-shaped domain Ω, parabolic inflow boundary condition, natural outflow boundary condition,



You can run the example as </> `E4-NavierStokes/navierstokes_solution.m`.

Navier-Stokes: backward facing step

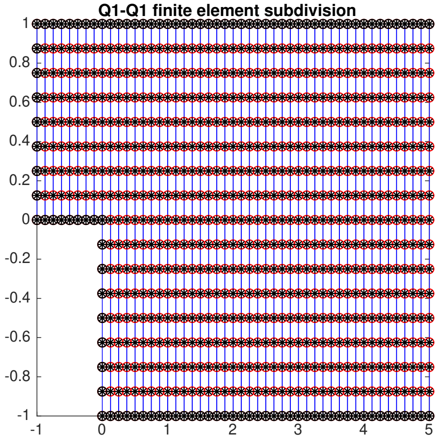
Test problem:

- L-shaped domain Ω, parabolic inflow boundary condition, natural outflow boundary condition,

Poiseuille flow

It is a steady horizontal flow in a channel driven by a pressure difference between the two ends

$$u_x = 1 - y^2,\ u_y = 0,\ p = -2\nu x + \text{constant}.$$

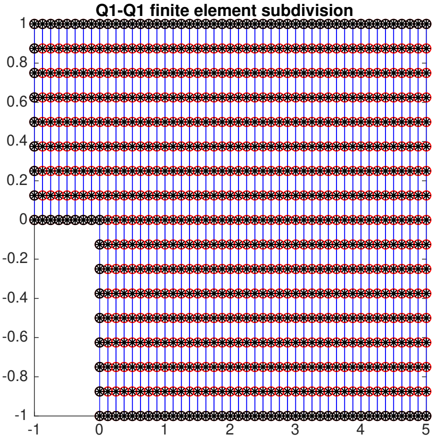


You can run the example as `E4-NavierStokes/navierstokes_solution.m`.

Navier-Stokes: backward facing step

Test problem:

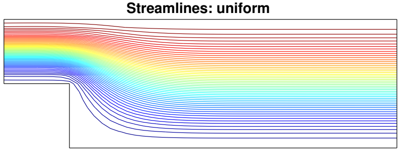
- L-shaped domain Ω, parabolic inflow boundary condition, natural outflow boundary condition,
- **Inflow** $x = -1$, $0 \leq y \leq 1$,
  No flow on the boundary,
  Neumann condition at the outflow $x = L$, $-1 < y < 1$.
- Discretized with (unstable) $Q1$-$Q1$ elements.

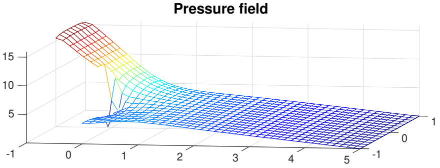


You can run the example as `E4-NavierStokes/navierstokes_solution.m`.

Navier-Stokes: backward facing step

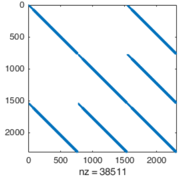
Initial guess:

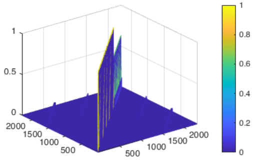




Solution of the associated Stokes problem.

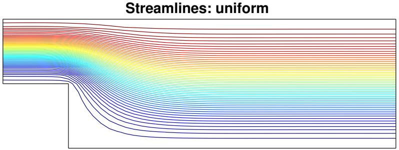
Picard's Iteration

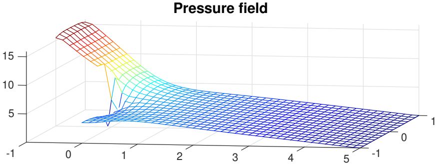




Navier-Stokes: backward facing step

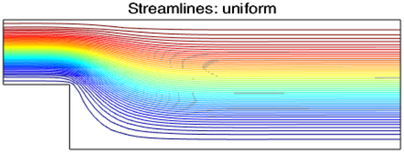
Initial guess:

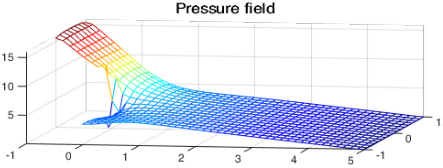




Solution of the associated Stokes problem.

Picard's Iteration

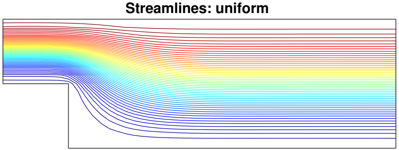


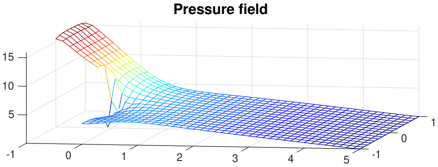


Iteration 1

Navier-Stokes: backward facing step

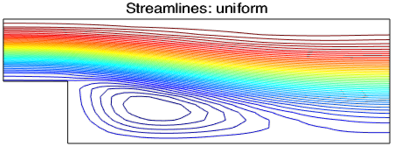
Initial guess:

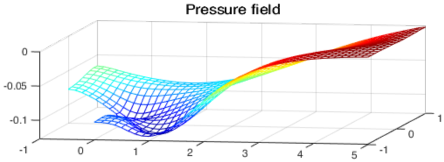




Solution of the associated Stokes problem.

Picard's Iteration

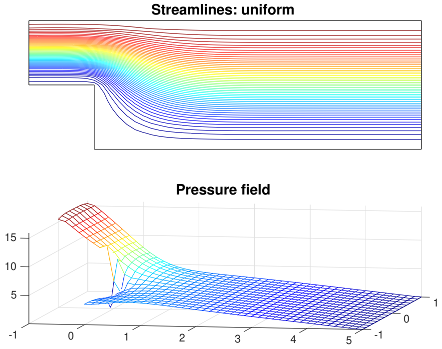




Iteration 2

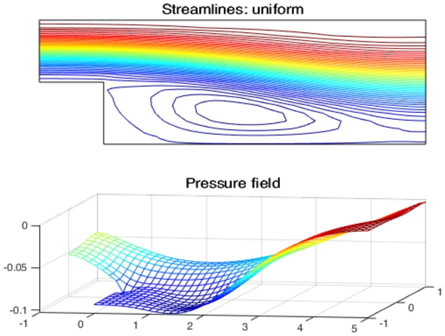
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

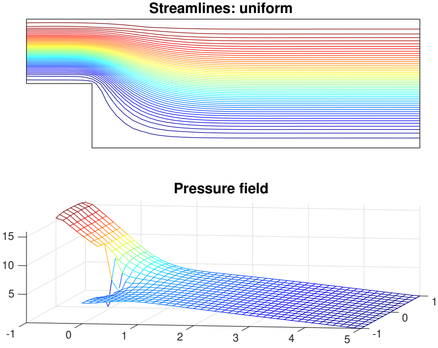
Picard's Iteration



Iteration 3

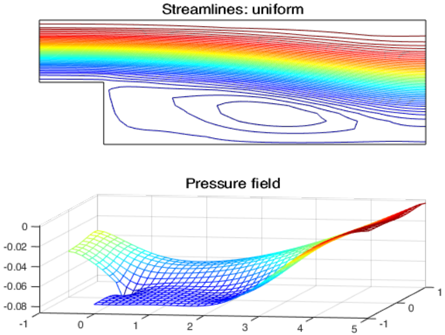
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

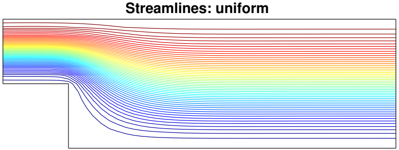
Picard's Iteration

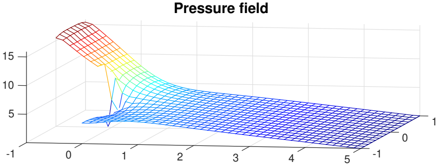


Iteration 4

Navier-Stokes: backward facing step

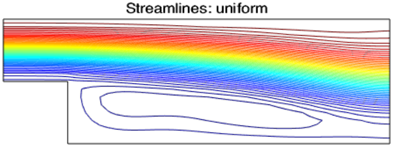
Initial guess:

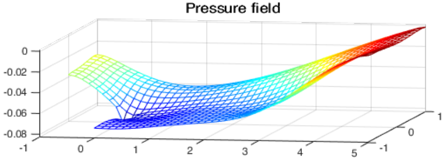




Solution of the associated Stokes problem.

Picard's Iteration

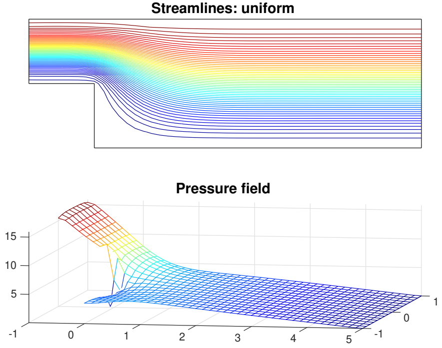




Iteration 5

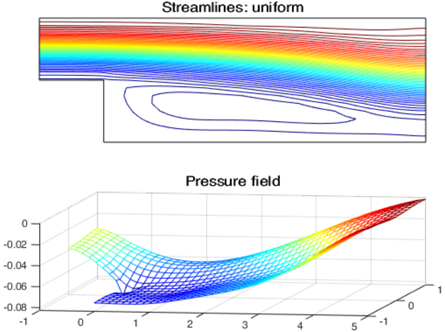
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

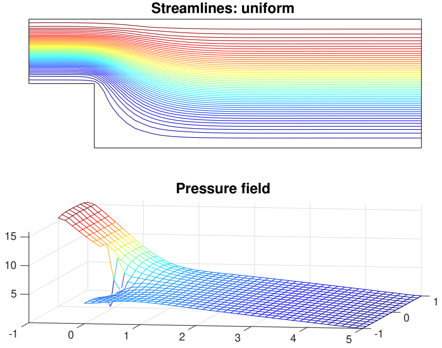
Picard's Iteration



Iteration 6

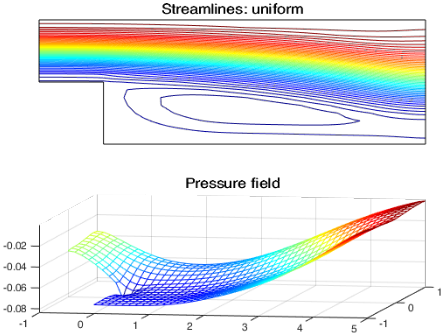
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

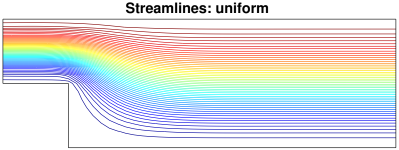
Picard's Iteration

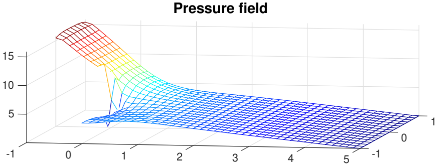


Iteration 7

Navier-Stokes: backward facing step

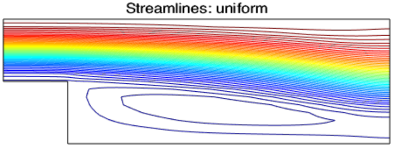
Initial guess:

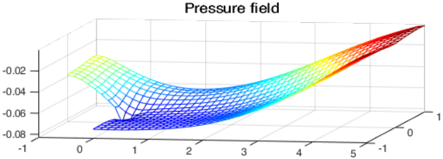




Solution of the associated Stokes problem.

Picard's Iteration

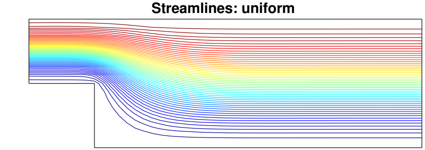


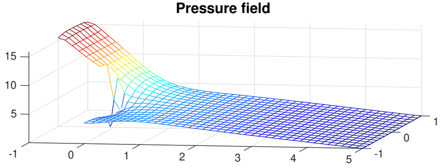


Iteration 8

Navier-Stokes: backward facing step

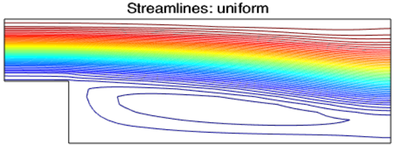
Initial guess:

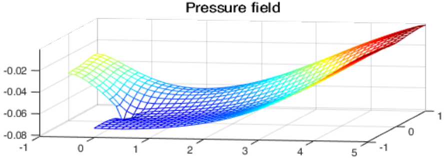




Solution of the associated Stokes problem.

Picard's Iteration

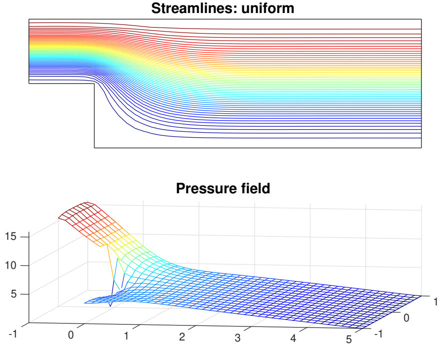




Iteration 9

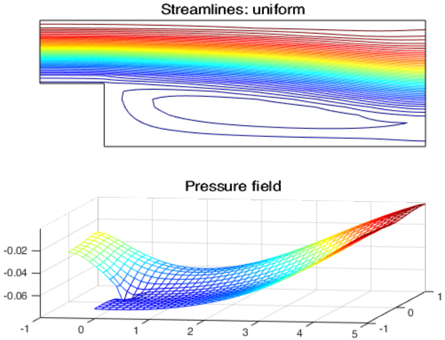
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

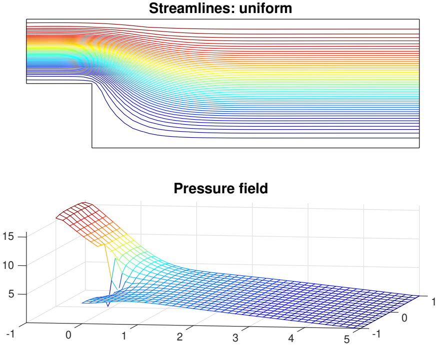
Picard's Iteration



Iteration 10

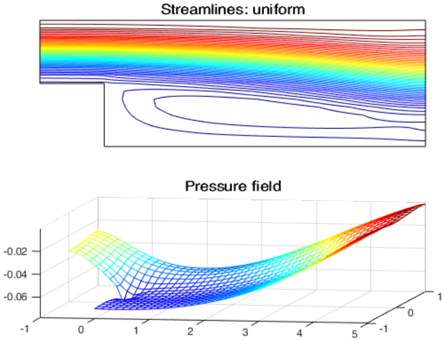
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

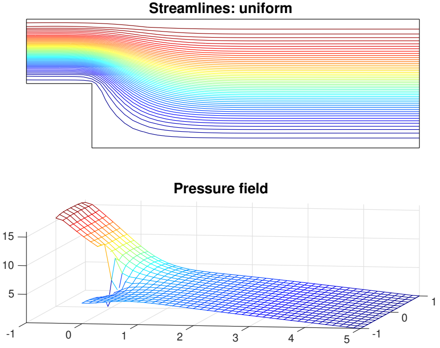
Picard's Iteration



Iteration 11

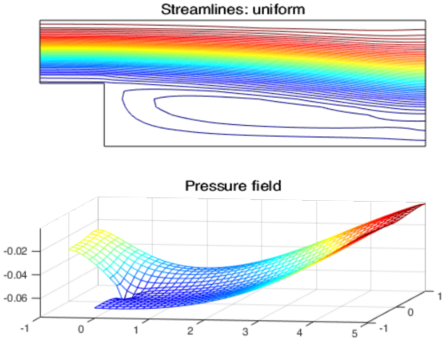
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

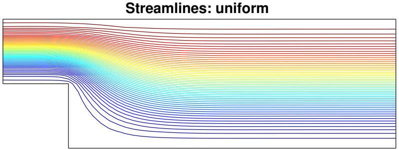
Picard's Iteration

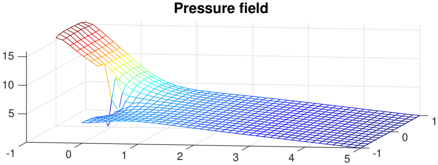


Iteration 12

Navier-Stokes: backward facing step

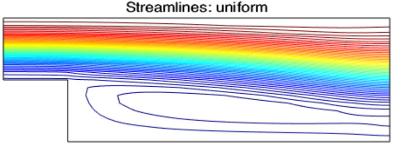
Initial guess:

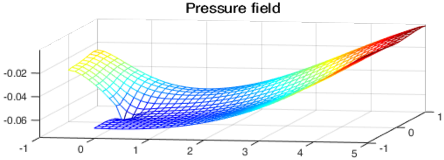




Solution of the associated Stokes problem.

Picard's Iteration

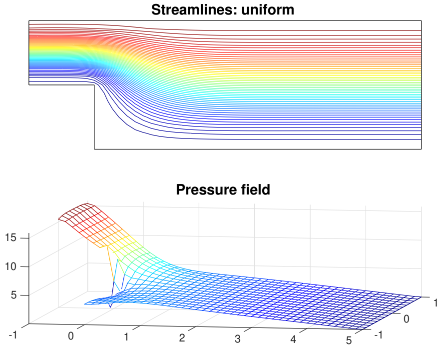




Iteration 13

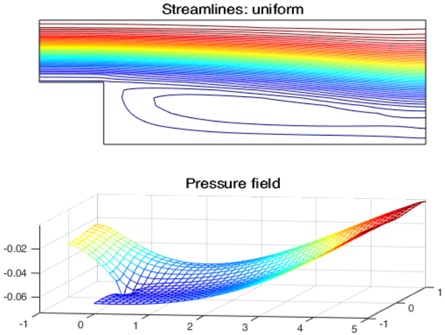
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

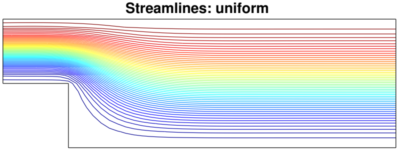
Picard's Iteration

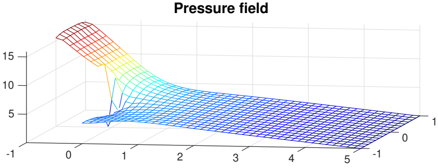


Iteration 14

Navier-Stokes: backward facing step

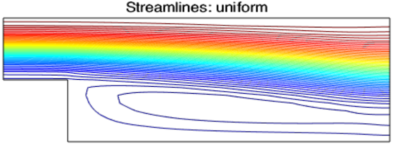
Initial guess:

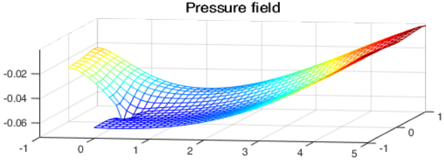




Solution of the associated Stokes problem.

Picard's Iteration

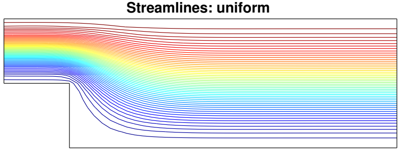


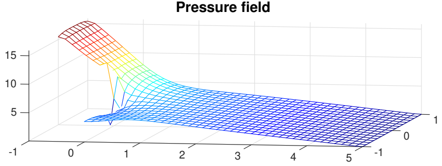


Iteration 15

Navier-Stokes: backward facing step

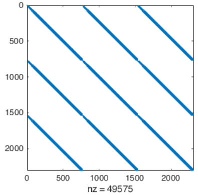
Initial guess:

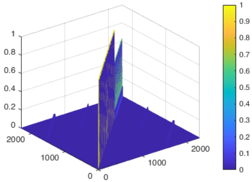




Solution of the associated Stokes problem.

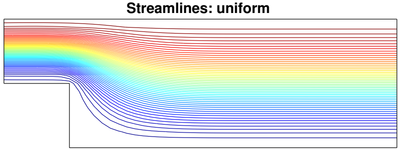
Newton Method

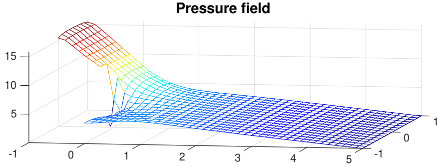




Navier-Stokes: backward facing step

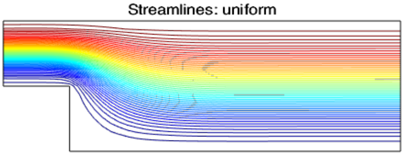
Initial guess:

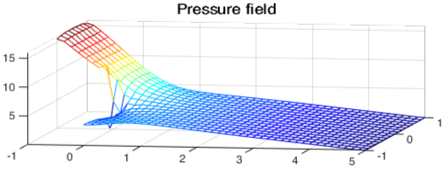




Solution of the associated Stokes problem.

Newton Method

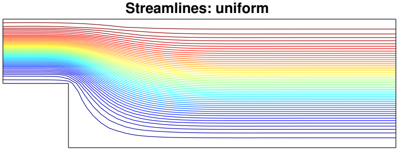


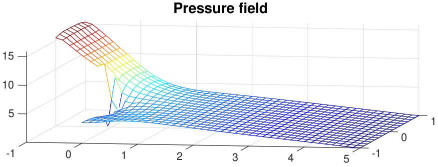


Iteration 1

Navier-Stokes: backward facing step

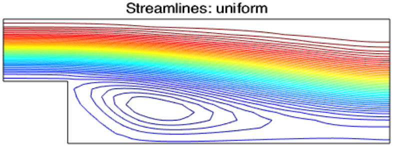
Initial guess:

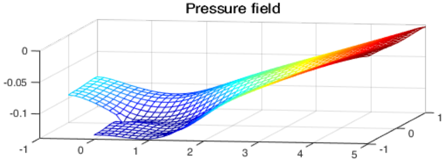




Solution of the associated Stokes problem.

Newton Method



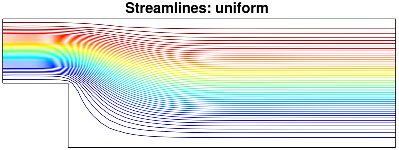


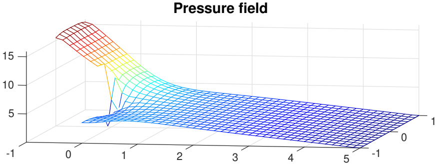
Iteration 2

Navier-Stokes: backward facing step

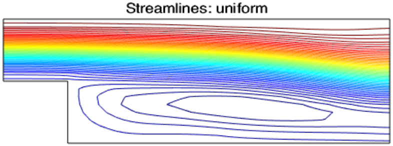
Initial guess:

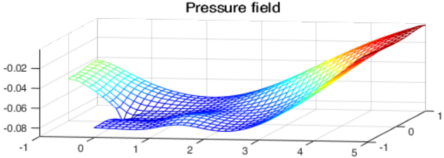
Newton Method





Solution of the associated Stokes problem.

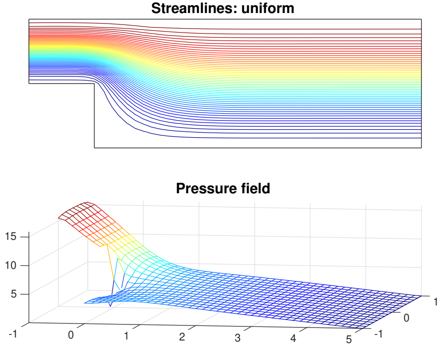




Iteration 3

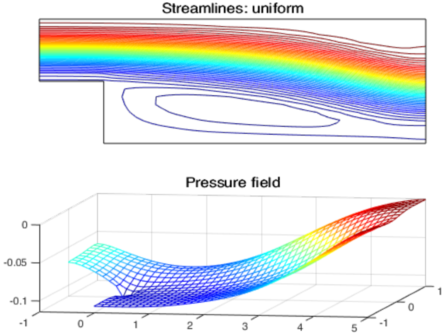
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

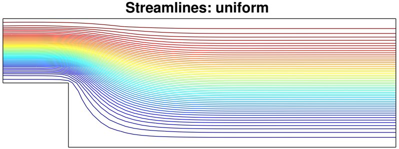
Newton Method

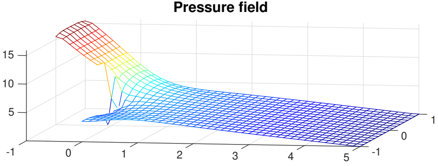


Iteration 4

Navier-Stokes: backward facing step

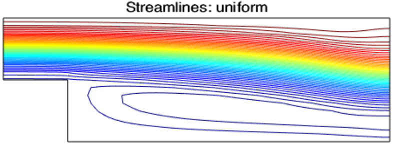
Initial guess:

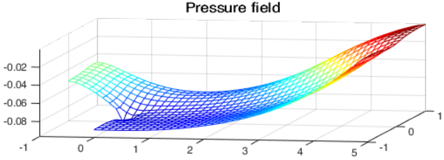




Solution of the associated Stokes problem.

Newton Method

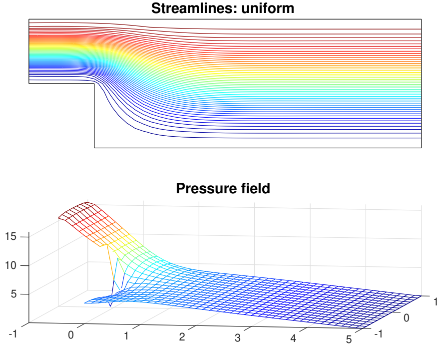




Iteration 5

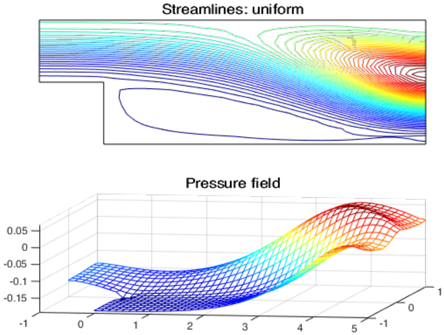
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

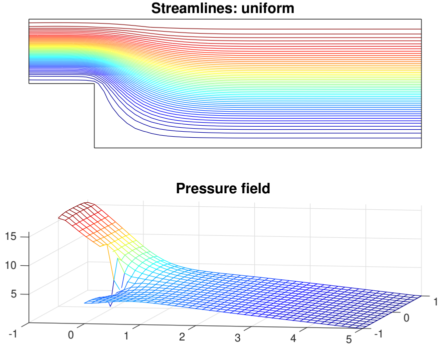
Newton Method



Iteration 6

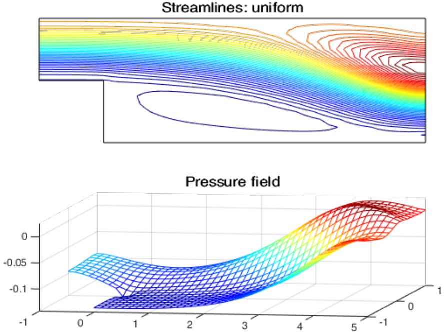
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

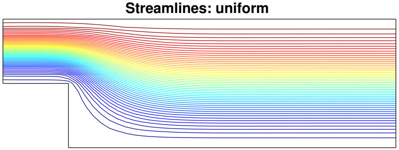
Newton Method

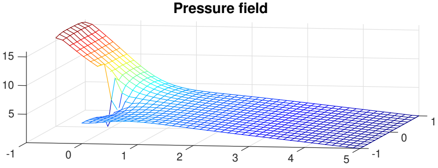


Iteration 7

Navier-Stokes: backward facing step

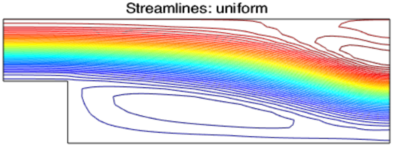
Initial guess:

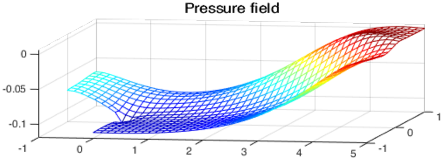




Solution of the associated Stokes problem.

Newton Method

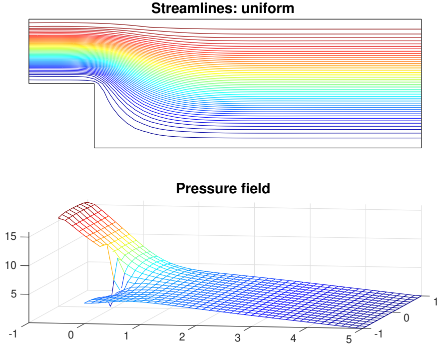




Iteration 8

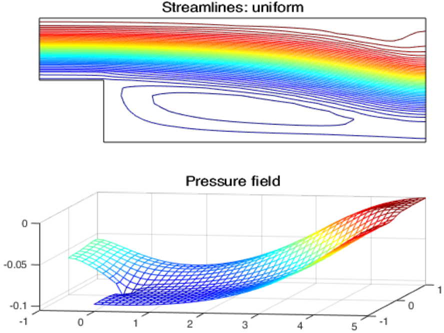
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

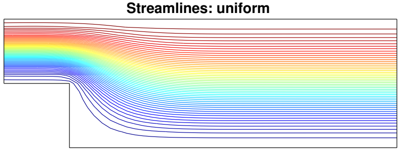
Newton Method

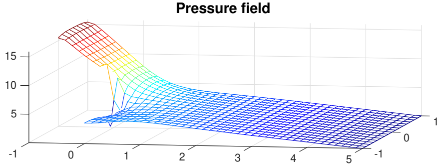


Iteration 9

Navier-Stokes: backward facing step

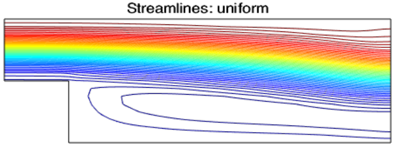
Initial guess:

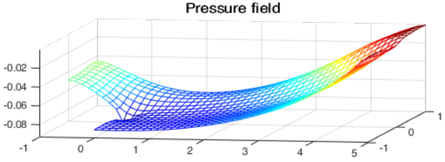




Solution of the associated Stokes problem.

Newton Method

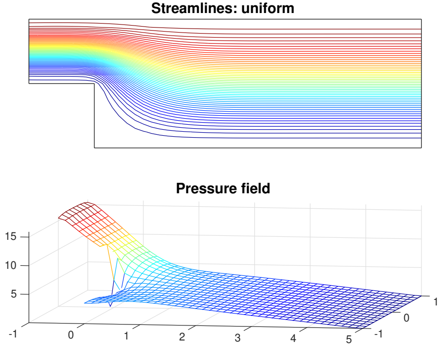




Iteration 10

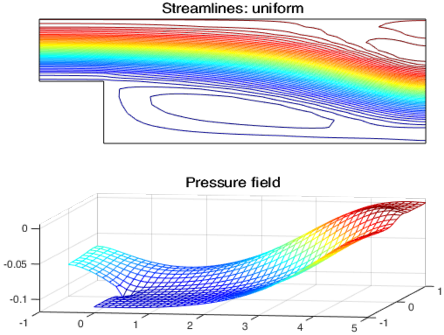
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

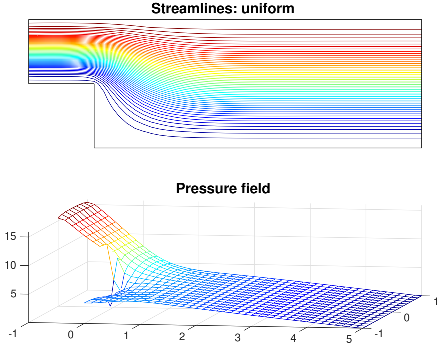
Newton Method



Iteration 11

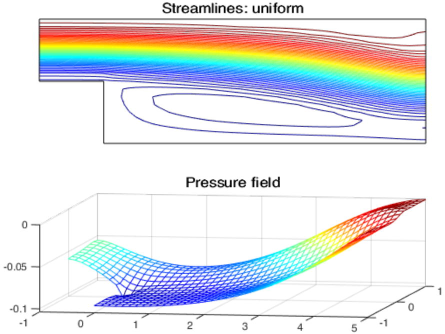
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

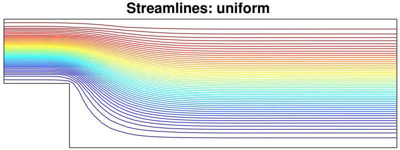
Newton Method

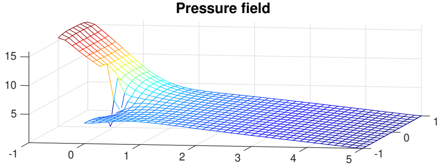


Iteration 12

Navier-Stokes: backward facing step

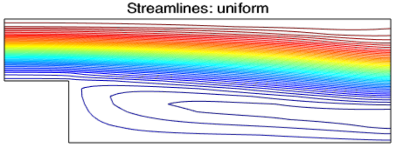
Initial guess:

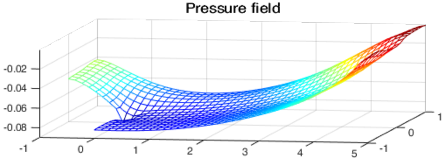




Solution of the associated Stokes problem.

Newton Method

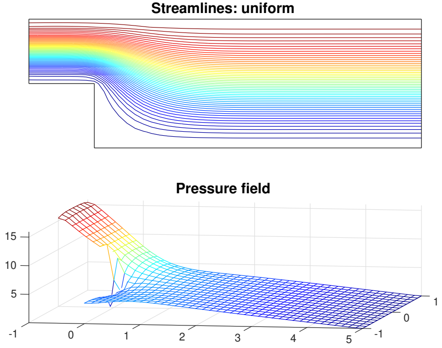




Iteration 13

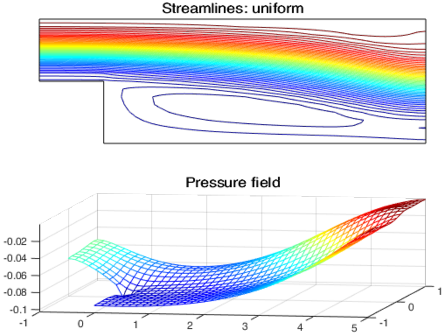
Navier-Stokes: backward facing step

Initial guess:



Solution of the associated Stokes problem.

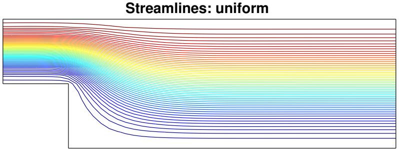
Newton Method

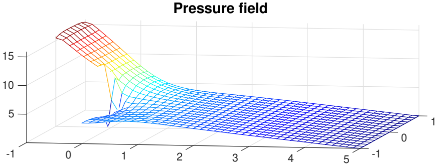


Iteration 14

Navier-Stokes: backward facing step

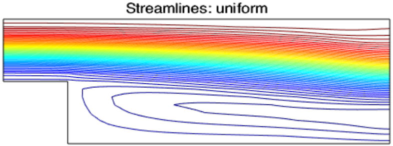
Initial guess:

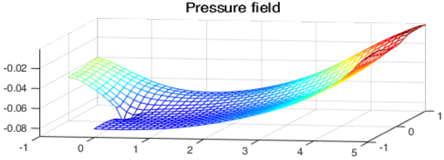




Solution of the associated Stokes problem.

Newton Method

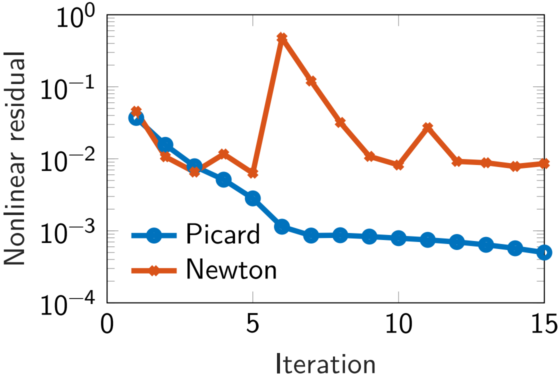




Iteration 15

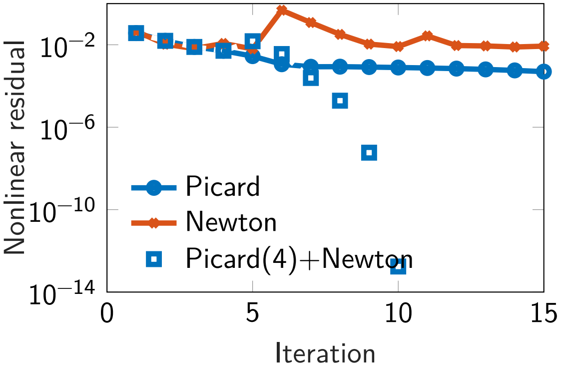
Navier-Stokes: backward facing step

- For this test problem convergence of the Newton method from the Stokes initial data is quite poor, what can we do?



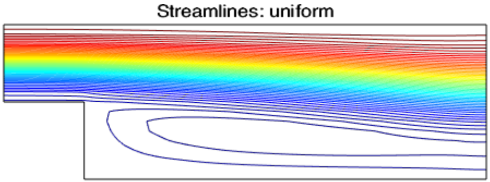
Navier-Stokes: backward facing step

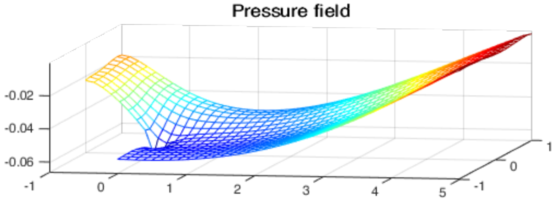
- For this test problem convergence of the Newton method from the Stokes initial data is quite poor, what can we do?
- We start from Stokes, then perform few steps of Picard's iteration, and finally *accelerate* with Newton.



Navier-Stokes: backward facing step

- For this test problem convergence of the Newton method from the Stokes initial data is quite poor, what can we do?
- We start from Stokes, then perform few steps of Picard's iteration, and finally *accelerate* with Newton.
- The "mess" doesn't end here – unfortunately or fortunately, I'm not yet sure...*boundary layers*, *bifurcations*, *absence of stable flows*,...





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