From Optimal Control to Saddle Point Matrices

Iterative Methods for Large-Scale Saddle-Point Problems

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"In the land of Mordor, in the fires of Mount Doom, the Dark Lord Sauron forged, in secret, a Master Ring to ***control all others****. And into this Ring he poured his cruelty, his malice and his will to dominate all life. One Ring to rule them all." – Galadriel*

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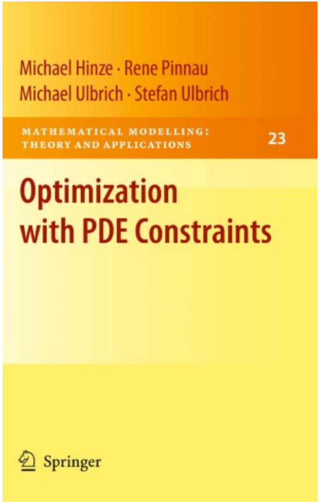
Whether you are *lawful good* or *lawful evil*, **control** is paramount.

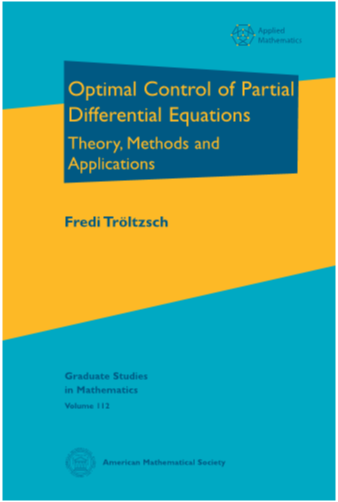
Overview

The main sources



M. Hinze et al. (2009). *Optimization with PDE constraints*. Vol. 23. Mathematical Modelling: Theory and Applications. Springer, New York, pp. xii+270. ISBN: 978-1-4020-8838-4

F. Tröltzsch (2010). *Optimal control of partial differential equations*. Vol. 112. Graduate Studies in Mathematics. Theory, methods and applications, Translated from the 2005 German original by Jürgen Sprekels. American Mathematical Society, Providence, RI, pp. xvi+399. ISBN: 978-0-8218-4904-0



Applications

When we know how to simulate a physical phenomenon, the next question we usually ask ourselves is: can we control it to benefit from it?

- Stationary problem of magnetohydrodynamics (Griesse and Kunisch 2006),
- Fluid-mechanics (Gunzburger and Manservisi 1999),
- Multi-phase flow in porous media (Hazra and Schulz 2005),
- Aerodynamic Shape Optimization (Jameson 1989),
- Nonlocal diffusion (Cipolla and Durastante 2018),
- ...

The abstract problem

We will consider problem of this form:

$$\min_{w \in W} J(w) \text{ subject to } e(w) = 0, \quad c(w) \in \mathcal{K}, \quad w \in \mathcal{C},$$

where

- $J : W \to \mathbb{R}$ is the objective function,
- $e : W \to Z$, and $c : W \to R$,
- W, Z and R are real Banach spaces,
- $\mathcal{K} \subset R$ is a closed convex cone,
- $\mathcal{C} \subset W$ is a closed convex set.

The (less) abstract problem

We can turn everything to the **finite-dimensional case**

$$\min_{w \in W} J(w) \text{ subject to } e(w) = 0, \quad c(w) \in \mathcal{K}, \quad w \in \mathcal{C},$$

where

$$W = \mathbb{R}^n, \quad Z = \mathbb{R}^l, \quad R = \mathbb{R}^m, \quad \mathcal{K} = (-\infty, 0]^m, \quad \mathcal{C} = \mathbb{R}^n.$$

Assuming

- J, c, e continuously differentiable,
- *Constraint qualification* (CQ)

KKT Conditions

Exist **Lagrange multipliers** $\overline{p} \in \mathbb{R}^l$, $\overline{\lambda} \in \mathbb{R}^m$ such that $(\overline{w}, \overline{p}, \overline{\lambda})$ solves

$$\begin{cases} \nabla J(\overline{w}) + c'(\overline{w})^T \overline{\lambda} + e'(\overline{w})^T \overline{p} = 0, \\ e(\overline{w}) = 0, \\ c(\overline{w}) \leq 0, \ \overline{\lambda} \geq 0, \ c(\overline{w})^T \overline{\lambda} = 0. \end{cases}$$

A stationary model problem

Let us consider the simplest problem we can formulate:

$$\min_{(y,u)\in Y\times U} J(y,u) = \frac{1}{2}\|y - z\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_U,$$

$$\text{s.t.} \begin{cases} Ay \equiv -\Delta y = Bu, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega, \\ u \in U_{\mathrm{ad}} \subseteq U, & y \in Y_{\mathrm{ad}} \subseteq Y. \end{cases} \qquad (\mathcal{P}_0)$$

- $\mathbb{R} \ni \alpha > 0$, $\Omega \subset \mathbb{R}^n$ a convex polyhedral domain,
- $Y = H^1_0(\Omega) = Y_{\mathrm{ad}}$,
  $B : U \to H^{-1}(\Omega) \equiv Y^*$,
- $U_{\mathrm{ad}} \subset U$ closed and convex.

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- $U_{\text{ad}} \subset U$ closed and convex.

Theorem: existence and uniqueness

Let $\alpha \geq 0$, $U_{\text{ad}} \subset U$ convex, closed and in the case $\alpha = 0$ bounded, $Y_{ad} \subset Y$ convex and closed, such that $(\mathcal{P}_0)$ has a feasible point, $\mathcal{A} \in \mathcal{L}(Y,Z)$ have a bounded inverse. Then $(\mathcal{P}_0)$ has an optimal solution $(\overline{y},\overline{u})$, moreover, if $\alpha > 0$ such solution is unique.

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- $U = \mathbb{L}^2(\Omega)$, $B : \mathbb{L}^2(\Omega) \to H^{-1}(\Omega)$ Injection,
- $U_{\mathrm{ad}} = \{v \in \mathbb{L}^2(\Omega) \ : \ a \leq v(x) \leq b \text{ a.e. } \Omega\}$, $a, b \in \mathbb{L}^\infty(\Omega)$

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- $U = \mathbb{R}^m$, $B : \mathbb{R}^m \to H^{-1}(\Omega)$ for $Bu = \sum_{j=1}^m u_j F_j$ and $F_j \in H^{-1}(\Omega)$ given,
- $U_{\mathrm{ad}} = \{\mathbf{v} \in \mathbb{R}^m : a_j \leq v_j \leq b_j\}$, $\mathbf{a} < \mathbf{b}$.

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The reduced problem

We need to derive the **KKT conditions** for a given problem, we proceed by steps

1. we reduce the problem to a minimization problem in the control u function,
2. we use the adjoint approach to derive the conditions we have to solve for.

The reduced problem

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1. we reduce the problem to a minimization problem in the control u function,
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We start again from:

$$\min_{\substack{y \in Y \\ u \in U}} J(y,u) \text{ subject to } e(y,u) = 0,\ (y,u) \in W_{\text{ad}} \subset W = Y \times U, \quad \begin{array}{l} e : Y \times U \to Z, \\ J : Y \times U \to \mathbb{R}, \end{array}$$

with W a nonempty closed subset of the product Banach space.

Definition

Let $F : U \subset X \to Y$ be an operator between Banach spaces X, Y and $U \neq \emptyset$ open

(a) F is called *directionally differentiable* at $x \in U$ if the limit

$$dF(x,h) = \lim_{t \to 0^+} \frac{F(x+th) - F(x)}{t} \in Y.$$

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Let $F : U \subset X \to Y$ be an operator between Banach spaces X, Y and $U \neq \emptyset$ open

(b) F is called *Gâteaux differentiable* at $x \in U$ if F is directionally differentiable at x and the directional derivative $F'(x) : X \ni h \mapsto dF(x,h) \in Y$ is **bounded** and **linear**, i.e., $F'(x) \in \mathcal{L}(X,Y)$.

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Definition

Let $F : U \subset X \to Y$ be an operator between Banach spaces X, Y and $U \neq \emptyset$ open

(c) F is called *Fréchet differentiable* at $x \in U$ if F is Gâteaux differentiable at x and if

$$\|F(x+h) - F(x) - F'(x)h\|_Y = o(\|h\|_X) \text{ for } \|h\|_X \to 0.$$

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$$\min_{\substack{y \in Y \\ u \in U}} J(y,u) \text{ subject to } e(y,u) = 0,\ (y,u) \in W_{\mathrm{ad}} \subset W = Y \times U,\ \begin{array}{l} e : Y \times U \to Z, \\ J : Y \times U \to \mathbb{R}, \end{array}$$

with W a nonempty closed subset of the product Banach space. We assume also that J and e are continuously *Fréchet differentiable* and that the **state equation** possesses for each $u \in U$ a unique corresponding solution $y(u) \in Y$:

$$\exists\, u \in U \mapsto y(u) \in Y, \quad e_y(y(u),u) \in \mathcal{L}(Y,Z) \text{ continuosly invertible.}$$

The reduced problem

Implicit Function Theorem

Let X, Y, Z be Banach spaces and let $F : G \to Z$ be a continuously *Fréchet differentiable* map from an open set $G \subset X \times Y \to Z$. Let $(\bar{x}, \bar{y}) \in G$ be such that $F(\bar{x}, \bar{y}) = 0$ and that $F_y(\bar{x}, \bar{y}) \in \mathcal{L}(Y, Z)$ has a bounded inverse.
Then there exists an open neighborhood $U_X(\bar{x}) \times U_Y(\bar{y}) \subset G$ of $(\bar{x}, \bar{y})$ and a unique continuous function $w : U_X(\bar{x})) \to Y$ such that

- $w(\bar{x}) = \bar{y}$,
- $\forall\, x \in U_x(\bar{x})\ \exists!\, y \in U_Y(\bar{y})$ with $F(x, y) = 0$, i.e., $y = w(x)$.

Moreover, the mapping $w : U_X(\bar{x}) \to Y$ is continuously *Fréchet differentiable* with derivative

$$w'(x) = F_y(x, w(x))^{-1} F_X(x, w(x)).$$

If $F : G \to Z$ is m-times continuously *Fréchet differentiable* then also $w\, U_X(\bar{x}) \to Y$ is m-times continuously *Fréchet differentiable*.

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$$\min_{\substack{y \in Y \\ u \in U}} J(y, u) \text{ subject to } e(y, u) = 0, \; (y, u) \in W_{\text{ad}} \subset W = Y \times U, \; \begin{array}{l} e : \; Y \times U \to Z, \\ J : \; Y \times U \to \mathbb{R}, \end{array}$$

with W a nonempty closed subset of the product Banach space. We assume also that J and e are continuously *Fréchet differentiable* and that the **state equation** possesses for each $u \in U$ a unique corresponding solution $y(u) \in Y$:

$$\exists \, u \in U \mapsto y(u) \in Y, \quad e_y(y(u), u) \in \mathcal{L}(Y, Z) \text{ continuosly invertible.}$$

Then the **Implicit Function Theorem** ensures that $y(u)$ is *continuously differentiable*.

The reduced problem

We need to derive the **KKT conditions** for a given problem, we proceed by steps

1. we reduce the problem to a minimization problem in the control u function,
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The reduced problem

An equation for the derivative of $y(u)$ is then obtained by

$$e_y(y(u),u)y'(u) + e_u(y(u),u) = 0,$$

and the reduced problem

$$\min_{u \in U} \hat{J}(u) = J(y(u),u) \text{ subject to } u \in \hat{U}_{\text{ad}} = \{u \in U \,:\, (y(u),u) \in W_{\text{ad}}\}.$$

The adjoint approach

We have now

$$\min_{u\in U} \hat{J}(u) = J(y(u),u) \text{ subject to } u \in \hat{U}_{\text{ad}} = \{u \in U \,:\, (y(u),u) \in W_{\text{ad}}\},$$
$$\text{with } e_y(y(u),u)y'(u) + e_u(y(u),u) = 0.$$

We now try to represent the derivative of $\hat{J}$

$$\begin{aligned} < \hat{J}'(u), s >_{U^*,U} &= < J_y(y(u),u), y'(u)s >_{Y^*,Y} + < J_u(y(u),u), s >_{U^*,U} \\ &= < y'(u)^* J_y(y(u),u), s >_{U^*,U} + < J_u(y(u),u), s >_{U^*,U}, \end{aligned}$$

and thus

$$\hat{J}'(u) = y'(u)^* J_y(y(u),u) + J_u(y(u),u)$$

The adjoint approach

We have now

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$$\text{with } e_y(y(u),u)y'(u) + e_u(y(u),u) = 0.$$

We represent the derivative of $\hat{J}$

$$\begin{aligned}\hat{J}'(u) &= y'(u)^* J_y(y(u),u) + J_u(y(u),u)\\ &= -\, e_u(y(u),u)^* e_y(y(u),u)^{-*} J_y(y(u),u) + J_u(y(u),u)\end{aligned}$$

The adjoint approach

We have now

$$\min_{u\in U} \hat{J}(u) = J(y(u),u) \text{ subject to } u \in \hat{U}_{\mathrm{ad}} = \{u \in U \,:\, (y(u),u) \in W_{\mathrm{ad}}\},$$

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where $p = p(u) \in Z^*$ is called the **adjoint state** and solves

$$e_y(y(u),u)^* p = -J_y(y(u),u). \qquad \text{(Adjoint Equation)}$$

The adjoint approach

We have now

$$\min_{u\in U} \hat{J}(u) = J(y(u),u) \text{ subject to } u \in \hat{U}_{\mathrm{ad}} = \{u \in U : (y(u),u) \in W_{\mathrm{ad}}\},$$
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We represent the derivative of $\hat{J}$

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We compute the derivative of $\hat{J}(u)$ by:

1. solving the adjoint equation for p,
2. and computing $\hat{J}'(u) = e_u(y(u),u)^* p + J_u(y(u),u)$.

The Linear–Quadratic Optimal Control Problem

Let's apply the theory developed till here to the following problem

$$\min_{(y,u)\in Y\times U} J(y,u) = \frac{1}{2}\|Qy - q_d\|_H^2 + \frac{\alpha}{2}\|u\|_U^2,$$
$$\text{subject to} \quad Ay + Bu = g, \quad u \in U_{\text{ad}},\ y \in Y_{\text{ad}},$$

where

- H, U are Hilbert spaces,
- Y, Z are Banach spaces,
- $q_d \in H$ is the **desired state**, $g \in Z$ is a source term,
- $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(Y,H)$.

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- We get the form for which we have proved both **Existence**, **Uniqueness**, and the **adjoint characterization**, by setting

$$e(y,u) = Ay + Bu - g, \quad W_{\text{ad}} = Y_{\text{ad}} \times U_{\text{ad}}.$$

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$$e(y,u) = Ay + Bu - g, \quad W_{\text{ad}} = Y_{\text{ad}} \times U_{\text{ad}}.$$

- The reduced problem is then obtained by means of the continuous affine linear solution operator: $U \ni u \mapsto y(u) = A^{-1}(g - Bu) \in Y$.

The Linear–Quadratic Optimal Control Problem

Let's apply the theory developed till here to the following problem

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For the derivatives we get:

$$\begin{aligned} < J_y(y,u), s_y >_{Y^*,Y} &= (Qy - q_d, Qs_y)_H = < Q^*(Qy - q_d), s_y >_{Y^*,Y} \\ < J_u(y,u), s_u >_{U^*,U} &= \alpha(u, s_u)_U, \\ e_y(y,u)s_y &= As_y, \\ e_u(y,u)s_u &= Bs_u. \end{aligned}$$

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We can thus recover the expressions for the operators

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\text{subject to} \quad & e(y,u) \equiv Ay + Bu - g = 0, \quad (y,u) \in W_{\text{ad}},
\end{aligned}
$$

We can thus recover the expressions for the operators

$$
\begin{aligned}
J_y(y,u) =& (Qy - q_d, Q\cdot)_H, \\
J_u(y,u) =& \alpha(u,\cdot)_U, \\
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We can thus recover the expressions for the operators

$$\begin{aligned}
J_y(y,u) =&(Qy - q_d, Q\cdot)_H =< (Qy - q_d), Q\cdot >_{H^*,H} =< Q^*(Qy - q_d), \cdot >_{Y^*,Y} \\
=&Q^*(Qy - q_d), \\
J_u(y,u) =&\alpha(u,\cdot)_U = \alpha u, \\
e_y(y,u) =&A, \\
e_u(y,u) =&B.
\end{aligned}$$

But U and H are Hilbert spaces, and Riesz says that:

"For every continuous linear functional $\phi \in H^*$ there exist a unique $f_\phi \in H$ such that $< f_\phi, x >_{H^*,H} = (x, f_\phi)_H$ for all $x \in H$".

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We can thus recover the expressions for the operators

$$\begin{aligned}J_y(y,u) =& Q^*(Qy - q_d),\\ J_u(y,u) =& \alpha u,\\ e_y(y,u) =& A,\\ e_u(y,u) =& B.\end{aligned}$$

The reduced functional is

$$\hat{J}(u) = J(y(u),u) = \frac{1}{2}\|Q(A^{-1}(g - Bu)) - q_d\|_H^2 + \frac{\alpha}{2}\|u\|_U^2.$$

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The reduced functional is

$$\hat{J}(u) = J(y(u),u) = \frac{1}{2}\|Q(A^{-1}(g - Bu)) - q_d\|_H^2 + \frac{\alpha}{2}\|u\|_U^2.$$

And we compute the derivative via the **adjoint approach**:

$$\text{Compute } p: \; A^*p = -(Qy - q_d, Q\cdot)_H, \quad \hat{J}'(u) = B^*p + \alpha(u,\cdot)_U$$

The Linear–Quadratic Optimal Control Problem

Let's apply the theory developed till here to the following problem

$$\begin{aligned} \min_{(y,u)\in Y\times U} J(y,u) = &\ \tfrac{1}{2}\|Qy - q_d\|_H^2 + \tfrac{\alpha}{2}\|u\|_U^2, \\ \text{subject to} &\ \ e(y,u) \equiv Ay + Bu - g = 0, \quad (y,u) \in W_{\text{ad}}, \end{aligned}$$

We can thus recover the expressions for the operators

$$\begin{aligned} J_y(y,u) =& Q^*(Qy - q_d), & e_y(y,u) =& A, \\ J_u(y,u) =& \alpha u, & e_u(y,u) =& B. \end{aligned}$$

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$$\text{Compute } p:\ A^*p = -Q^*(Qy - q_d), \quad \hat{J}'(u) = B^*p + \alpha u.$$

Optimality conditions

After all this manipulation we have managed rewriting our problem in the form

$$\min_{w \in W} J(w) \text{ subject to } w \in \mathcal{C} \text{ with } \begin{array}{ll} W & \text{Banach,} \\ J : W \to \mathbb{R} & \text{Gâteaux differentiable,} \\ \mathcal{C} \subset W \neq \emptyset, & \text{closed and convex.} \end{array} \qquad (\ast)$$

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Theorem

Il J is defined on an open neighborhood of $\mathcal{C}$, and $\bar{u}$ is a local solution of $(\boldsymbol{*})$ at which J is Gâteaux differentiable. Then the following optimality condition holds:

$$\bar{w} \in \mathcal{C}, \ < J'(\bar{w}), w - \bar{w} >_{W^*,W} \geq 0 \ \forall \, w \in \mathcal{C}.$$

If J is **convex** on $\mathcal{C}$ the condition is **necessary** and **sufficient** for global optimality. If it is *strictly convex* then there exists at most one solution.

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If W is reflexive, $\mathcal{C}$ is closed and convex, and J is convex and continuous with

$$\lim_{\substack{w \in \mathcal{C} \\ \|w\|_W \to \infty}} J(w) = +\infty,$$

then there exist a global solution of the problem.

Optimality conditions in the original notation

Let us roll-back to the original formulation and summarize all the conditions

$$\min_{(y,u)\in Y\times U} J(y,u) \text{ subject to } e(y,u)=0, \qquad u \in U_{\mathsf{ad}}.$$

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Assumptions

- $\emptyset \neq U_{\text{ad}} \subset U$ convex, and closed,
- $J : Y \times U \to \mathbb{R}$, $e : Y \times U \to \mathbb{R}$, continuously differential on Banach spaces U, Y, Z,
- $\forall\, u \in V \subset U$ open neighborhood of U_{ad} the state equation has a unique solution,
- $e_y(y(u),u) \in \mathcal{L}(Y,Z)$ has a bounded inverse $\forall u \in V \supset U_{\text{ad}}$.

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Theorem

If the **assumptions** hold and $\bar{u}$ is a local solution of the reduced problem

$$\min_{u\in U} \hat{J}(u) = J(y(u),u) \text{ s.t. } u\in U_{\text{ad}}$$

then $\bar{u}$ satisfies the variational inequality

$$\bar{u}\in U_{\text{ad}} \text{ and } <\hat{J}'(u), u-\hat{u}>_{U^*,U} \geq 0,\ \forall\, u\in U_{\text{ad}}.$$

Optimality conditions Linear-Quadratic Problem

We conclude the parabola by looking at the conditions for the Linear-Quadratic Problem

$$\min_{(y,u)\in Y\times U} J(y,u) = \frac{1}{2}\|Qy - q_d\|_H^2 + \frac{\alpha}{2}\|u\|_U^2,$$
$$\text{subject to } Ay + Bu = g, \quad u \in U_{\text{ad}}$$

that take the form

$$\begin{aligned} A\bar{y} + B\bar{u} &= g, && \text{State Equation} \\ A^*\bar{p} &= -Q^*(Q\bar{y} - q_d), && \text{Adjoint Equation} \\ \bar{u} \in U_{\text{ad}}, \quad (\alpha\bar{u} + B^*\bar{p}, u - \bar{u}) &\geq 0, \quad \forall u \in U_{\text{ad}}. && \text{Variational Inequality} \end{aligned}$$

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Where are we?

Okay, but after all this mountain of calculations, where is the saddle-point matrix?

The distributed control of elliptic equations

We finally have all the machinery in place to approach the simplest problem

$$\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathbb{L}^2(\Omega)},$$

$$\text{subject to } \begin{cases} -\Delta y = \beta u, & \text{on } \Omega, \\ y = 0, & \text{on } \partial\Omega \end{cases}$$

This is an instance of the **Linear-Quadratic Problem** in which we have dropped the bound on control u, i.e., $U_{\text{ad}} \equiv U \equiv \mathbb{L}^2(\Omega)$.

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We move to the *weak formulation* of the constraint

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This is an instance of the **Linear-Quadratic Problem** in which we have dropped the bound on control u, i.e., $U_{\text{ad}} \equiv U \equiv \mathbb{L}^2(\Omega)$.

We move to the *weak formulation* of the constraint, and find the first two conditions:

$$\int_\Omega \nabla y \cdot \nabla v \,\mathrm{d}x - \int_\Omega \beta u v \,\mathrm{d}x = 0, \quad \forall\, v \in \mathbb{H}^1_0(\Omega), \qquad \text{State equation}$$

$$\int_\Omega \nabla p \cdot \nabla w \,\mathrm{d}x + \int_\Omega (y - y_d) w \,\mathrm{d}w = 0, \quad \forall\, w \in \mathbb{H}^1_0(\Omega). \qquad \text{Adjoint equation}$$

Where we have used the fact that the bilinear for the elliptic equation, i.e., the operator A of the general formulation, is self-adjoint.

The distributed control of elliptic equations

To complete

$$\int_\Omega \nabla y \cdot \nabla v \, \mathrm{d}x - \int_\Omega \beta u v \, \mathrm{d}x = 0, \quad \forall\, v \in \mathbb{H}_0^1(\Omega), \qquad \text{State equation}$$

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we need the *variational inequality*

$$\bar{u} \in U_{\mathrm{ad}}, \quad (\alpha \bar{u} + B^* \bar{p}, u - \bar{u}) \geq 0, \quad \forall u \in U_{\mathrm{ad}}.$$

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we need the *variational inequality*, in which we first observe that B^* is indeed the product by $-\beta$ and $U_{\mathrm{ad}} \equiv \mathbb{L}^2(\Omega)$

$$\bar{u} \in \mathbb{L}^2(\Omega), \quad (\alpha \bar{u} - \beta \bar{p}, u - \bar{u}) \geq 0, \quad \forall u \in \mathbb{L}^2(\Omega).$$

The distributed control of elliptic equations

To complete

$$\int_\Omega \nabla y \cdot \nabla v \,\mathrm{d}x - \int_\Omega \beta u v \,\mathrm{d}x = 0, \quad \forall\, v \in \mathbb{H}^1_0(\Omega), \qquad \text{State equation}$$

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we need the *variational inequality*, in which we first observe that B^* is indeed the product by $-\beta$ and $U_{\text{ad}} \equiv \mathbb{L}^2(\Omega)$, therefore this is indeed an equality

$$\alpha u - \beta p = 0. \qquad a.e.$$

The distributed control of elliptic equations

The conditions are then

$$\begin{aligned} \int_\Omega \nabla y \cdot \nabla v \,\mathrm{d}x - \int_\Omega \beta u v \,\mathrm{d}x = 0, &\quad \forall\, v \in \mathbb{H}^1_0(\Omega), && \text{State equation} \\ \int_\Omega \nabla p \cdot \nabla w \,\mathrm{d}x + \int_\Omega (y - y_d) w \,\mathrm{d}x = 0, &\quad \forall\, w \in \mathbb{H}^1_0(\Omega). && \text{Adjoint equation} \\ \alpha u - \beta p = 0. &\qquad a.e. && \text{Gradient condition} \end{aligned}$$

We can now use our expertise on finite elements to discretize the three conditions obtaining

$$\begin{array}{r} \text{Adjoint equation} \\ \text{Gradient condition} \\ \text{State equation} \end{array} \begin{bmatrix} M & O & A \\ O & \alpha M & -\beta M \\ A & -\beta M & O \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M\mathbf{y}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \begin{array}{l} m_{i,j} = \int_\Omega \phi_i \phi_j, \\ a_{i,j} = \int_\Omega \nabla \phi_i \cdot \nabla \phi_j, \end{array}$$

A better look at the saddle

We now reduced ourselves to the problem

$$\begin{bmatrix} M & O & A \\ 0 & \alpha M & -\beta M \\ A & -\beta M & O \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M\mathbf{y}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

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- We have selected the same FEM space both for u and y, this is *not compulsory*, but permits us to rewrite the system in a simpler form:

$$\alpha M\mathbf{u} = \beta M\mathbf{p} \Rightarrow \mathbf{u} = \frac{\beta}{\alpha}\mathbf{p} \Rightarrow \begin{bmatrix} M & A \\ A & -\frac{\beta^2}{\alpha} M \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M\mathbf{y}_d \\ \mathbf{0} \end{bmatrix}$$

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- Nevertheless both system have the same Schur complement

$$[A - \beta M] \begin{bmatrix} M^{-1} & O \\ O & \frac{1}{\alpha} M^{-1} \end{bmatrix} \begin{bmatrix} A \\ -\beta M \end{bmatrix} = AM^{-1}A + \frac{\beta^2}{\alpha} M$$

A better look at the saddle

We now reduced ourselves to the problem

$$\begin{bmatrix} M & O & A \\ 0 & \alpha M & -\beta M \\ A & -\beta M & O \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M\mathbf{y}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- To discuss the behavior of lower/upper bound, and thus of the ill-conditioning one needs information on the actual FEM space used, for P_1-Q_1 elements on quasi-uniform grids we can investigate an $\inf$-$\sup$ condition

$$\begin{aligned} \min_{\mathbf{p}} \frac{< (AM^{-1}A + \frac{\beta^2}{\alpha}M)\mathbf{p}, \mathbf{p} >}{< A\mathbf{p}, \mathbf{p} >} \geq \min_{\mathbf{p}} \frac{< AM^{-1}\mathbf{p}, \mathbf{p} >}{< A\mathbf{p}, \mathbf{p} >} = \min_{\mathbf{w} = A\mathbf{p}} \frac{< M^{-1}\mathbf{w}, \mathbf{w} >}{< A^{-1}\mathbf{w}, \mathbf{w} >} \\ = \min_{\mathbf{w} = A\mathbf{p}} \frac{< A\mathbf{w}, \mathbf{w} >}{< M\mathbf{w}, \mathbf{w} >} \geq c_\Omega, \end{aligned}$$

for c_Ω the Poincare constant, so the saddle-point system is “$\inf$-$\sup$ stable”,

A better look at the saddle

We now reduced ourselves to the problem

$$\begin{bmatrix} M & O & A \\ 0 & \alpha M & -\beta M \\ A & -\beta M & O \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M\mathbf{y}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- analogously for the largest eigenvalue

$$\max_{\mathbf{p}} \frac{< (AM^{-1}A + \frac{\beta^2}{\alpha} M)\mathbf{p}, \mathbf{p} >}{< A\mathbf{p}, \mathbf{p} >} \geq \begin{cases} \max_{\mathbf{w}} {}^{<A\mathbf{w},\mathbf{w}>}/_{<M\mathbf{w},\mathbf{w}>}, \\ \frac{\beta^2}{\alpha} \max_{\mathbf{p}} {}^{<M\mathbf{p},\mathbf{p}>}/_{<A\mathbf{p},\mathbf{p}>}. \end{cases}$$

Approximating the Schur complement

The matrix A is then far from being a good preconditioner for the Schur complement!

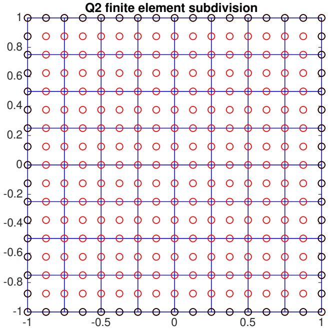
An example

- Domain $\Omega = [-1, 1]^2$,
- Desired state:

$$y_d = \begin{cases} x_1^2 x_2^2, & \text{in } \Omega_1 = [-1, 0]^2, \\ 0, & \text{in } \Omega \setminus \Omega_1 \end{cases}$$

- We fix the Dirichlet boundary to match the desired state.

We discretize with Q_2-elements:



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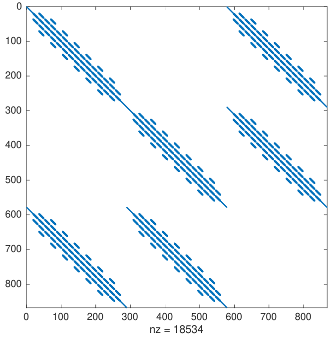
We discretize with Q_2-elements. Obtaining the saddle-point matrix:



An example

Finding the solution:

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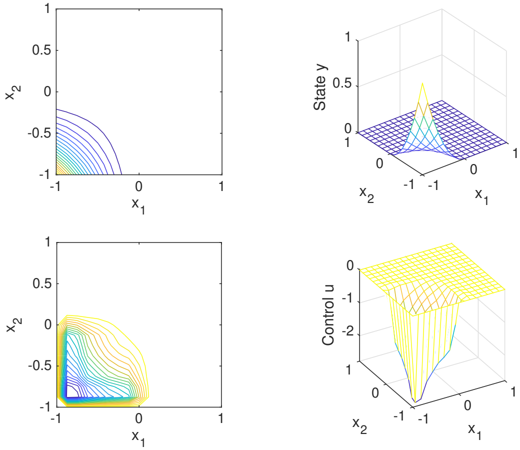
- We fix the Dirichlet boundary to match the desired state.
- Fixing $\alpha = 10^{-5}$, $\beta = 1$ we solve.



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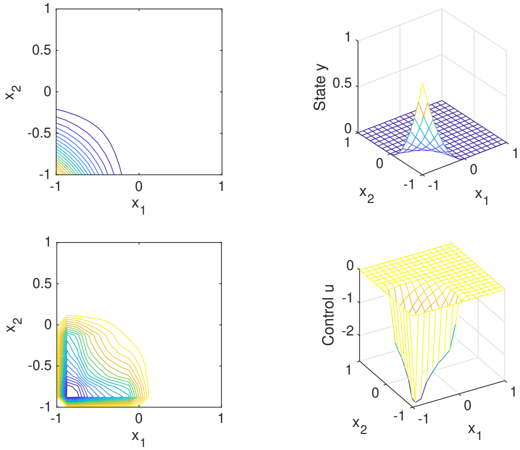

Code for the example: `E5-OptimalPoisson/examplepoisson_control.m`

Boxed constraints

Bounds on the control

In applications usually a "**control function**" costs something or has some type of natural constraints, e.g., we are controlling a phenomena of *chemotaxis* and u is the amount of medication, it has to stay in between a certain level before becoming toxic and a level that is the minimum effective dosage; or if it is a mechanical controller it cannot exert forces at every possible intensity.

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When we work with $U = \mathbb{L}^2(\Omega)$, one of the most used form of this bound is given by

$$U_{\text{ad}} = \{u \in U \;:\; a \leq u \leq b \text{ a.e. } a \leq b\} \text{ for } a, b \in \mathbb{L}^2(\Omega).$$

This type of limits are called **box constraints**.

Boxed constraints

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This type of limits are called **box constraints**.
To use them *in practice* we need to find a way to rewrite the *variational inequality* in a more manageable way:

$$\bar{u} \in U_{\text{ad}} \text{ and } < \hat{J}'(u), u - \hat{u} >_{U^*,U} \geq 0, \; \forall\, u \in U_{\text{ad}}.$$

Boxed constraints

Proposition

Let $U \in \mathbb{L}^2(\Omega)$, $a, b \in \mathbb{L}^2$, $a \leq b$, and U_{ad} be given by

$$U_{\text{ad}} = \{u \in U \;:\; a \leq u \leq b \text{ a.e. } \}.$$

Then the following conditions are equivalent:

(i) $\bar{u} \in U_{\text{ad}}$ $(\nabla \hat{J}(\bar{u}), u - \bar{u})_U \geq 0 \;\forall\, u \in U_{\text{ad}}$,

(ii) $\bar{u} \in U_{\text{ad}}$ $\nabla \hat{J}(\bar{u}) \begin{cases} = 0, & \text{if } a(x) < \bar{u}(x) < b(x), \\ \geq 0, & \text{if } a(x) = \bar{u}(x) < b(x), \\ \leq 0, & \text{if } a(x) < \bar{u}(x) = b(x), \end{cases}$ for a.a. $x \in \Omega$.

(iii) There $\bar{\lambda}_a$, $\bar{\lambda}_b \in U^* = \mathbb{L}^2(\Omega)$ with

$$\begin{aligned} & & & \nabla \hat{J}(\bar{u}) + \bar{\lambda}_b - \bar{\lambda}_a = 0, \\ \bar{u} &\geq a, & \bar{\lambda}_a &\geq 0, & \bar{\lambda}_a(\bar{u} - a) &= 0, \\ \bar{u} &\leq b, & \bar{\lambda}_b &\geq 0, & \bar{\lambda}_b(b - \bar{u}) &= 0. \end{aligned}$$

The distributed control of elliptic equations

$$\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathbb{L}^2(\Omega)},$$

$$\text{subject to } \begin{cases} -\Delta y = \beta u, & \text{on } \Omega, \\ y = 0, & \text{on } \partial\Omega \end{cases}$$

$$a \leq u \leq b \qquad \text{on } \Omega.$$

The distributed control of elliptic equations

$$
\begin{aligned}
&\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathbb{L}^2(\Omega)}, \\
&\text{subject to } \int_\Omega \nabla y \cdot \nabla v \,\mathrm{d}x - \int_\Omega \beta u v \,\mathrm{d}x = 0, \qquad \forall v \in \mathbb{H}^1_0(\Omega). \\
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$$a \le u \le b \qquad \text{on } \Omega.$$

And use the new characterization for the variational inequality

$$\int_\Omega \nabla y \cdot \nabla v \,\mathrm{d}x - \int_\Omega \beta u v \,\mathrm{d}x = 0, \quad \forall\, v \in \mathbb{H}^1_0(\Omega), \qquad \text{State equation}$$

$$\int_\Omega \nabla p \cdot \nabla w \,\mathrm{d}x + \int_\Omega (y - y_d) w \,\mathrm{d}x = 0, \quad \forall\, w \in \mathbb{H}^1_0(\Omega). \qquad \text{Adjoint equation}$$

$$\begin{aligned} \alpha \bar{u} - \gamma \bar{p} + \bar{\lambda}_b - \bar{\lambda}_a &= 0, \\ \bar{u} \ge a, \quad \bar{\lambda}_a \ge 0, \quad \bar{\lambda}_a(\bar{u} - a) &= 0, \\ \bar{u} \le b, \quad \bar{\lambda}_b \ge 0, \quad \bar{\lambda}_b(b - \bar{u}) &= 0, \end{aligned} \qquad \text{Gradient condition}$$

The distributed control of elliptic equations

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$$(x_1, x_2) \in \mathbb{R}^2 \quad x_1 \geq 0,\ x_2 \geq 0, \quad x_1 x_2 = 0,\ \Leftrightarrow\ \phi(x_1, x_2) = 0,$$

  with $\phi : \mathbb{R}^2 \to \mathbb{R}$.

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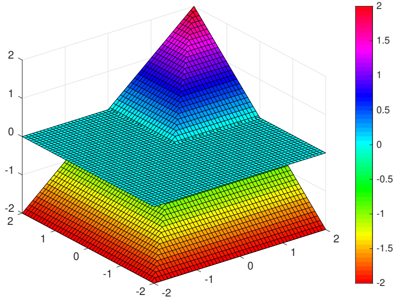
  with $\phi : \mathbb{R}^2 \to \mathbb{R}$.

An example of such function is indeed

$$\phi(x_1, x_2) = \min\{x_1, x_2\}$$

that satisfies our request **but** is not globally differentiable.



Idea

Variational inequalities $\Leftrightarrow$ nonsmooth equations.

Semismooth optimization: projected gradient

For our problem we introduce the *projection*

$$P_s(w)(x) = P_{[a(x),b(x)]}(w(x)) = max(a(x), min(w(x), b(x))),$$

and rewrite the *gradient condition* as

$$\Psi(w) = w - P_s(w - \theta J'(w)) = 0, \quad \theta \in \mathbb{R}_+$$

To solve our **optimization problem** we need to find a suitable substitution for G'.

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Semismoothness

Let $\Psi : X \to Y$ be a continuous operator between Banach spaces. Furthermore, let $\partial\Psi : X \rightrightarrows Y$ be a **set valued** mapping with nonempty images, then

- Ψ is called $\partial\Psi$-semismooth at $x \in X$ if

$$\sup_{M \in \partial\Psi(x+d)} \|\Psi(x+d) - \Psi(x) - Md\|_Y = o(\|d\|_X), \quad \text{for } \|d\|_X \to 0.$$

Semismooth optimization: projected gradient

Clarke's generalized derivative

$\Psi(x) = P_{[a,b]}(x) : \mathbb{R} \to \mathbb{R}$, $a < b$, admits generalized derivative

$$\partial^{\mathrm{cl}}\psi(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ 1, & a < x < b, \\ [0,1], & a = a \text{ or } x = b. \end{cases}$$

Semismooth optimization: projected gradient

Clarke's generalized derivative

$\Psi(x) = P_{[a,b]}(x) : \mathbb{R} \to \mathbb{R}$, $a < b$, admits generalized derivative $\partial\psi^{\mathrm{cl}}(x)$.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be measurable with $0 < |\Omega| < \infty$, $\Psi : \mathbb{R}^m \to \mathbb{R}$ Lipschitz continuous and semismooth. Let Y be a Banach space, $1 \leq q < p \leq \infty$, and assume that $G : Y \to \mathbb{L}^q(\Omega)^m$ is continuously F-differentiable and that G maps Y locally Lipschitz continuously to $\mathbb{L}^p(\Omega)$, then the operator

$$\psi_G : Y \to \mathbb{L}^q(\Omega),\ \psi_G(y)(x) = \psi(G(y))(x)$$

is $\partial\Psi$-semismooth, with

$$\partial\Psi_G(y) = \{M : \ Mv = g^T(G'(y)v), \quad g \in \mathbb{L}^\infty(\Omega)^m, \quad g(x) \in \partial^{\mathrm{cl}}\psi(G(y)(x)) \text{ for a.a. } x \in \Omega$$

Semismooth Newton

With this construction we can build an extension of the classical Newton method

Input: Semismooth function G
Choose $x^0 \in X$
for $k = 0, 1, 2, \ldots$ **do**
  Chose $M_k \in \partial G(x^k)$
  Obtain s^k by solving

$$M_k s^k = -G(x^k)$$

  Set $x^{k+1} = x^k + s^k$.
end

Theorem

Let $G : X \to Y$ be continuous and ∂G-semismooth at a solution of $G(x) = 0$. We assume that

$$\exists\, C, \delta > 0 \,:\, \|M^{-1}\|_{Y \to X} \le C \quad \begin{array}{c} \forall M \in \partial G(x)\, \forall\, x \in X \\ \text{s.t. } \|x - \bar{x}\|_X < \delta, \end{array}$$

at a solution $\bar{x}$. Then, for all $x^0 \in X$, $\|x^0 - \bar{x}\|_X < \delta$ the semismooth Newton method converges to $\bar{x}$ superlinearly.

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The whole theory is quite laborious to develop, if you are interested a good starting point is the book Ulbrich 2011.

A worked out example

Since we are going to use Newton method, we modify our problem to a **semilinear one**

$$\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathbb{L}^2(\Omega)},$$

$$\text{subject to } \begin{cases} -\nabla^2 y + y^3 = 0 & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

$$u \leq u_b$$

We use the *weak-formulation* to derive the **optimality conditions**:

$$\int_\Omega \nabla y \cdot \nabla v + \int_\Omega y^3 \nabla v - \int_\Omega uv = 0, \qquad \forall v \in \mathbb{H}^1_0(\Omega), \qquad \text{State equation}$$

$$\int_\Omega \nabla p \cdot \nabla w + 3\int_\Omega y^2 pw - \int_\Omega (y - y_d)w, \qquad \forall w \in \mathbb{H}^1_0(\Omega), \qquad \text{Adjoint equation}$$

$$\alpha u + p + \lambda = 0, \quad \text{a.a. } x \in \Omega \qquad \text{Gradient condition}$$

$$\lambda - \max\{0, \lambda + \alpha(u - u_b)\} = 0 \quad \text{a.a. } x \in \Omega.$$

😈 A worked out example: details

- We need to **prove existence** for the solution of the *semilinear equation*, as we have seen for the Navier-Stokes problem this is a difficult issue in general. This equation is way simpler since it is of **monotone type**.

Theorem (Minty–Browder)

Let V be a separable, reflexive Banach space, then the variational equation

$$< A(y), v >_{V',V} = < \ell, v >_{V',V} \qquad \forall\, v \in V,$$

with

(i) Monotone A, i.e., $\forall\, u, v \in V < A(u) - A(v), u - v >_{V',V} \geq 0$,

(ii) Hemicontinuous, i.e., $t \mapsto < A(u + tv), w >_{V',V}$ is in $\mathcal{C}^0([0,1])$ $\forall, u, v, w \in V$,

(iii) Coercive, ${<A(u),u>_{V',V}}/{\|u\|_V} \overset{\|u\|_V \to \infty}{\longrightarrow} +\infty$,

admits a solution. Furthermore, if A is **strictly monotone**, then such solution is unique.

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- We need to **prove existence** for the solution of the *semilinear equation*, as we have seen for the Navier-Stokes problem this is a difficult issue in general. This equation is way simpler since it is of **monotone type**.
- We need to express the *generalized derivative* for the last equation, the mapping $\max(0,\cdot) : \mathbb{L}^q(\Omega) \to \mathbb{L}^p(\Omega)$, $1 \leq p < q \leq +\infty$ admits a generalized derivative of the form
$$G_{\max}(0, y) = \begin{cases} 1, & y(x) > 0, \\ 0, & y(x) \leq 0 \end{cases} : \mathbb{L}^q(\Omega) \to \mathcal{L}(\mathbb{L}^q(\Omega), \mathbb{L}^p(\Omega)).$$

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- To compute this generalized derivatives we need the values of the function y in the nodes, if we work by using FEM spaces, we are solving fo the coefficients in the basis expansion. This means that we have to use interpolation (or Lagrangian schemes) to obtain the desired results. To simplify the discussion, we ***go back to the strong formulation and use finite differences***.

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\alpha u + p + \lambda = 0, &\qquad \text{a.a. } x \in \Omega &\qquad \text{Gradient condition} \\
\lambda - \max\{0, \lambda + \alpha(u-u_b)\} = 0 &\qquad \text{a.a. } x \in \Omega. &
\end{aligned}$$

A worked out example: strong formulation

Our conditions in strong form read as

$$\begin{cases} -\nabla^2 y + y^3 - u = 0, & x \in \Omega \\ y = 0, & x \in \partial\Omega \end{cases}$$

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- If we call L the Finite Difference discretization of the Laplacian operator on Ω with 0 Dirichlet BCs then

$$F(\mathbf{y}, \mathbf{u}, \mathbf{p}, \boldsymbol{\lambda}) = \begin{bmatrix} L\mathbf{y} + \mathbf{y}^3 - \mathbf{u} \\ L^T\mathbf{p} + 3\mathbf{y}^2\mathbf{p} - (\mathbf{y} - \mathbf{y}_d) \\ \alpha\mathbf{u} + \mathbf{p} + \lambda \\ \boldsymbol{\lambda} - \max\{0, \boldsymbol{\lambda} + \alpha(\mathbf{u} - \mathbf{u}_b)\} \end{bmatrix}$$

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Then the Jacobian is given by

$$J_F = \begin{bmatrix} L + 3Y^2 & -I & O & O \\ -I + 6YP & 0 & L + 3Y^2 & O \\ O & \alpha I & I & I \\ O & -\alpha\chi & O & I - \chi \end{bmatrix}$$

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where

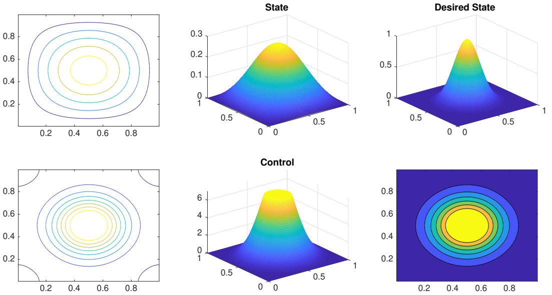
$$\chi = \operatorname{diag}(\chi_i), \quad \chi_i = \mathbf{1}_{\{\lambda_i + \alpha(u_i - u_b)\}}$$

A worked out example

Let us select the **desired state**

$$y_d(x,y) = \exp(-30((x - 1/2)^2 + (y - 1/2)^2)),$$

and the values $\alpha = 10^{-3}$ and $u_b = 7$.




```
Y=spdiags(y,0,n^2,n^2);
P=spdiags(p,0,n^2,n^2);
Xi=spdiags(spones(max(0,lam+alpha*(u-ubvec))),
↪   0,n^2,n^2);
A=[L+3*Y.^2 -I 0 0
-I+6*Y.*P 0 L+3*Y.^2 0
0 alpha*I I I
0 -alpha*Xi 0 I-Xi];
F=[ -L*y-y.^3+u
-L*p-3*Y.^2*p+y-yd
-p-alpha*u-lam
-lam+max(0,lam+alpha*(u-ubvec))];
```

The example can be run with the code in

</> `E5-OptimalPoisson/boundedcontrol_fd.m`

Characterization via the reduced gradient

We rewrite our condition and use the semismooth idea to rewrite the conditions for

$$\min_{\substack{y \in \mathbb{H}_0^1(\Omega) \\ u \in \mathbb{L}^2(\Omega)}} J(y,u) = \frac{1}{2}\|y - y_d\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathbb{L}^2(\Omega)} \text{ s.t. } Ay = r + Bu, \; a \le u \le b$$

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1. We eliminate the state $y = y(u) = A^{-1}(r + Bu)$,
2. For the **reduced gradient** we obtain

$$(\nabla\hat{J}(u),d) = (y(u)-y_d, y'(u)d)_{\mathbb{L}^2(\Omega)} + a(u,d)_{\mathbb{L}^2(\Omega)} = (y'(u)^*(y(u)-y_d)+\alpha u, d)_{\mathbb{L}^2(\Omega)}$$

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$$\nabla \hat{J}(u) = y'(u)^*(y(u)-y_d)+\alpha u = B^*(A^{-1}) * (A^{-1}(r+Bu)-y_d)+\alpha u = \alpha u + H(u).$$

3. $B \in \mathcal{L}(L^{p'}(\Omega), \mathbb{H}^{-1}(\Omega)$, $B^* \in \mathcal{L}(\mathbb{H}^1_0(\Omega), \mathbb{L}^p(\Omega)) \Rightarrow H(u)$ is a continuous, linear, affine mapping between $\mathbb{L}^2(\Omega) \to \mathbb{L}^p(\Omega)$.

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4. We rewrite then the gradient condition as
$$\Phi(u) = u - P_{[a,b]}(-1/\alpha H(u)) = 0.$$

Characterization via the reduced gradient

The **Newton system** for

$$\min_{u \in \mathbb{L}^2(\Omega)} J(y(u), u) = \frac{1}{2}\|y(u) - y_d\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathbb{L}^2(\Omega)} \text{ s.t. } \quad a \le u \le b$$

is then given by

$$\left(I + \frac{1}{\alpha}\partial^{\mathrm{cl}}\Phi(-{}^1\!/\!_\alpha H(u^k))H'(u^k)\right) s^k = -\Phi(u^k)$$

where $\partial^{\mathrm{cl}}\Phi(\cdot)H'(\cdot)$ is a short-hand for $v \mapsto \partial^{\mathrm{cl}}\Phi(\cdot) \cdot (H'(\cdot)v)$ and $\partial^{\mathrm{cl}}\Phi(\cdot)$ is Clarke's generalized derivative.

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$$(I + \frac{1}{\alpha}\partial\Phi(u^k)\cdot B^*A^{-*}A^{-1}B)s^k = -\Phi(u^k),$$

Characterization via the reduced gradient

The **Newton system** for

$$\min_{u\in\mathbb{L}^2(\Omega)} J(y(u),u) = \frac{1}{2}\|y(u)-y_d\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathbb{L}^2(\Omega)} \text{ s.t. } \quad a \le u \le b$$

is then given by

$$\left(I + \frac{1}{\alpha}\partial^{\mathrm{cl}}\Phi(-1/\alpha H(u^k))H'(u^k)\right)s^k = -\Phi(u^k)$$

where $\partial^{\mathrm{cl}}\Phi(\cdot)H'(\cdot)$ is a short-hand for $v \mapsto \partial^{\mathrm{cl}}\Phi(\cdot)\cdot(H'(\cdot)v)$ and $\partial^{\mathrm{cl}}\Phi(\cdot)$ is Clarke's generalized derivative. In operator terms we can rewrite it as

$$\begin{bmatrix} I & O & A^* \\ O & I & -\frac{1}{\alpha}\partial\Phi(u^k)\cdot B^* \\ A & -B & O \end{bmatrix}\begin{bmatrix} d_y^k \\ d_u^k \\ d_p^k \end{bmatrix} = \begin{bmatrix} O \\ -\Phi(u^k) \\ 0 \end{bmatrix}, \qquad s^k \equiv d_u^k.$$

Sparsity constraints

To conclude this gallery of optimization problems we consider the last case given by problem with **sparsity constraints**:

$$\min_{(y,u)} J(y,u) + \beta\|u\|_1 \text{ s.t. } e(y,u) = 0.$$

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Sparsity promoting

Let $\mathbf{x} = (1, \varepsilon) \in \mathbb{R}^2$, $0 < \varepsilon << 1$, then $\|\mathbf{x}\|_1 = 1 + \varepsilon$, and $\|\mathbf{x}\|_2^2 = 1 + \varepsilon^2$. An optimization process reduces the magnitude of one of the two entries by $0 < \delta < \varepsilon$, then

$$\mathbf{x}^{(1)} = \mathbf{x} - (\delta, 0) \Rightarrow \|\mathbf{x}^{(1)}\|_p^p = \begin{cases} 1 - \delta + \varepsilon, & p = 1, \\ 1 - 2\delta + \delta^2 + \varepsilon^2, & p = 2 \end{cases}$$

or

$$\mathbf{x}^{(1)} = \mathbf{x} - (0, \delta) \Rightarrow \|\mathbf{x}^{(1)}\|_p^p = \begin{cases} 1 - \delta + \varepsilon, & p = 1, \\ 1 - 2\delta\epsilon + \delta^2 + \varepsilon^2, & p = 2 \end{cases}$$

For ℓ_2 reducing x_1 does much more than reducing x_2, putting things to zero has diminishing returns with respect to reducing “large entries”. **For ℓ_1 it is exactly the same**.

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- As we always do we look at the reduced functional

$$\min_{u \in U} J(y(u),u) + \beta \|u\|_{\mathbb{L}^1(\Omega)}$$

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- This is globally *non smooth*, **but** it is obtained as the sum of a *regular part* and of a *convex* non differentiable function.

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Theorem

Let U be a Banach space, $j_1 : U \to \mathbb{R}$ Gâteaux differentiable, $j_2 : U \to \mathbb{R} \cup \{+\infty\}$ convex and continuous. If $\bar{u}$ is a *locally optimal* solution to $\min_{u\in U} j_1(u) + j_2(u)$, then it satisfies the variational inequality

$$j_1'(\bar{u}) + j_2(v) - j_2(\bar{u}) \geq 0, \quad \forall\, v \in U.$$

Sparsity constraints: optimality conditions

Let us focus on our favorite problem

$$\min_{(y,u)\in \mathbb{H}_0^1(\Omega)\times\mathbb{L}^2(\Omega)} J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2 + \frac{\alpha}{2}\|u\|^2_{\mathbb{L}^2(\Omega)} + \beta\|u\|_{\mathbb{L}^1(\Omega)},$$
$$\text{subject to } e(y,u) = 0.$$

If we apply the previous Theorem we get the variational inequality:

$$(y(\bar{u}) - y_d, y'(\bar{u})(v-\bar{u})) + \alpha(\bar{u}, v-\bar{u}) + \beta\|v\|_{\mathbb{L}^1(\Omega)} - \beta\|\bar{u}\|_{\mathbb{L}^1(\Omega)} \geq 0, \quad \forall\, v \in \mathbb{L}^2(\Omega).$$

This an example of an **Elliptic Variational Inequality of the Second Kind**, see (Glowinski 2008, Chapter 1.6), and there exist a way to rewrite them to a form to which we can apply the semismooth idea through the use of a *penalty function*.

Sparsity constraints: optimality conditions

By a **penalty argument** we can prove the existence of a $\lambda \in \mathbb{L}^2(\Omega)$ such that the condition

$$(y(\bar{u}) - y_d, y'(\bar{u})(v - \bar{u})) + \alpha(\bar{u}, v - \bar{u}) + \beta\|v\|_{\mathbb{L}^1(\Omega)} - \beta\|\bar{u}\|_{\mathbb{L}^1(\Omega)} \geq 0, \quad \forall\, v \in \mathbb{L}^2(\Omega).$$

is equivalent to

$$\begin{aligned}
&(y(\bar{u}) - y_d, y'(u)v) + (\alpha\bar{u} + \lambda, v) = 0, && \forall\, v \in U,\\
&\lambda = \beta, && \text{in } \{x \in \Omega \,:\, \bar{u} > 0\},\\
&|\lambda| \leq \beta, && \text{in } \{x \in \Omega \,:\, \bar{u} = 0\},\\
&\lambda = -\beta, && \text{in } \{x \in \Omega \,:\, \bar{u} < 0\},
\end{aligned}$$

Sparsity constraints: optimality conditions

By a **penalty argument** we can prove the existence of a $\lambda \in \mathbb{L}^2(\Omega)$ such that the condition

$$(y(\bar{u}) - y_d, y'(\bar{u})(v - \bar{u})) + \alpha(\bar{u}, v - \bar{u}) + \beta\|v\|_{\mathbb{L}^1(\Omega)} - \beta\|\bar{u}\|_{\mathbb{L}^1(\Omega)} \geq 0, \quad \forall\, v \in \mathbb{L}^2(\Omega).$$

And permits to write the **complete set of optimality conditions** for any $\theta > 0$:

$$e(\bar{y}, \bar{u}) = 0, \qquad \text{State equation}$$

$$e_y(\bar{y}, \bar{u})^* p = \bar{y} - y_d, \qquad \text{Adjoint equation}$$

$$\begin{cases} p + \alpha\bar{u} + \lambda = 0, \\ \bar{u} - \max\{0, \bar{u} + \theta(\lambda - \beta)\} - \min\{0, \bar{u} + \theta(\lambda + \beta)\} = 0. \end{cases} \qquad \text{Gradient condition}$$

Semismooth Newton

We already now how to "derive" the $\max$ function, the minimum is completely analogous

$$G_{\min}\{0, v\}(x) = \begin{cases} 0, & \text{if } v(x) \geq 0, \\ 1, & \text{if } v(x) < 0. \end{cases}$$

A worked out example

We reuse our **semilinear problem**

$$\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathbb{L}^2(\Omega)} + \beta\|u\|_{\mathbb{L}^1(\Omega)},$$

$$\text{subject to } \begin{cases} -\nabla^2 y + y^3 = 0 & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

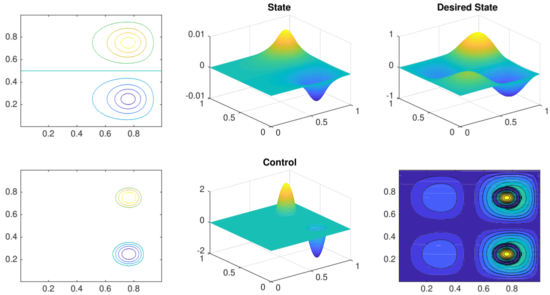
We state again the **optimality conditions** in strong form:

$$\begin{cases} -\nabla^2 y + y^3 - u = 0, & x \in \Omega \\ y = 0, & x \in \partial\Omega \end{cases} \qquad \begin{cases} -\nabla^2 p + 3y^2 p - (y - y_d) = 0, & x \in \Omega \\ p = 0, & x \in \partial\Omega \end{cases}$$

$$\begin{cases} p + \alpha u + \lambda = 0, \\ u - \max\{0, u + {}^1\!/\!_\alpha(\lambda - \beta)\} \\ \quad - \min\{0, u + {}^1\!/\!_\alpha(\lambda + \beta)\} = 0. \end{cases}$$

A worked out example

Select the **desired state** is $y_d(x, y) = \sin(2\pi x)\sin(2\pi y)\exp(2x)/6$, and the values $\alpha = 10^{-3}$ and $u_b = 0.008$.




```
Y=spdiags(y,0,n^2,n^2);P=spdiags(p,0,n^2,n^2);
Ximax=spdiags(spones(max(0,u+1/alpha*(lam-beta))),0,n^2,n^2);
Ximin=spdiags(spones(min(0,u+1/alpha*(lam+beta))),0,n^2,n^2);
Xi=Ximax+Ximin;
A=[L+3*Y.^2 -I O O
-I+6*Y.*P O L+3*Y.^2 O
O alpha*I I I
O I-Xi O -1/alpha*Xi];
F=[ -L*y-y.^3+u
-L*p-3*Y.^2*p+y-yd
-p-alpha*u-lam
-u+max(0,u+1/alpha*(lam-beta))+min(0,u+1/alpha*(lam+beta))];
```

The example can be run with the code in: </> `E5-OptimalPoisson/l1control_fd.m`

The rest of the world

What are we leaving out?

- **Non-stationary problems:** in many case we want to control the time-evolution of a problem, e.g.,

$$\begin{aligned} \min_{y,u} J(y,u) = & \frac{1}{2}\|y(T,x) - y_T(x)\|^2 + \frac{1}{2}\alpha T \int_\Omega \int_0^T y\|u\|^2, \\ \text{subject to } & u_t + \nabla \cdot (uy) = 0, \\ & u(0,x) = u_0. \end{aligned}$$

- **Other black-box methods:** interior point methods, active sets, decomposition methods (ADMM),...
- **Coupled problems:** there are many cases in which we are interested in phenomena that are described by the coupling of ODEs and PDEs.

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